MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

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DR. J. H. JEANS, PRESIDENT, in the Chair.

The President read the following resolution of the Council :—

"The Council of the Royal Astronomical Society, deeply conscious of the great services rendered to astronomy in general and to the Society in particular by their late President, Dr. Dreyer, wish to express their sense of loss at his death, and offer their condolences to his family in their bereavement."

William Barron Coutts, M.A., B.Sc., F.R.S.E., 74 Vanbrugh Park, Blackheath, London, S.E. 3 ;
Idwal Owain Griffith, M.A., 37 Banbury Road, Oxford ;
Mohammed Ali Higab, M.Sc., Ph.D., The University, Leeds; and Kanayat, Zagazig, Egypt ;
Charles Hardiman Laver, B.A., B.M., B.Ch., F.R.C.S., The Red House, Redhill, Surrey ;
Mrs. Dorothy Una Ratcliffe, The Hall, Roundhay, Leeds ; and
Leo Mengersen Waterford, Acting Chief Assistant, Adelaide Observatory, South Australia,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Norman Knight Niness, 65 Douglas Road, Handsworth, Birmingham (proposed by D. W. Horner) ;

Thomas Whitwell, Government Actuary's Dept., Treasury Chambers, Whitehall, S.W. 1 ; and 42 Seymour Road, Hornsey, London, N. 8 (proposed by L. J. Comrie) ; and
Captain John Williams, The Moorings, Aberdovey, N. Wales (proposed by J. Silas Evans).

Two hundred and sixteen presents were announced as having been received since the last meeting, including, amongst others :—

D. Baxandall, Calculating Machines and Instruments : Catalogue of the Collections in the Science Museum, S. Kensington (presented by the Director, Science Museum).
Bordeaux, Observatoire : Catalogue Photographique du Ciel, tome v. zones $+12°$à$+14°$.
Tychonis Brahe Dani Opera Omnia, tomus xiii. (presented by the representatives of the late Dr. Dreyer).
Catania, Reale Osservatorio, Catalogo Astrofotografico, vol. ii., pt. 2.
Cordoba, Observatorio Nacional Argentino, Catalogo Astrográfico, zona $-25°$ [Resultados, vol. xxvii.].
Evolution in the Light of Modern Knowledge : a collective work (presented by Dr. Jeans).
B. Fessenkoff, Photometrical Catalogue of 1155 stars.
R. Jarry-Desloges, Observations des Surfaces Planétaires, fasc. 7 (1919, 1920).
Johannesburg, Union Observatory, Southern Double Star Catalogue, parts 1–4 ($2^h\ 57^m.7$–$3^h\ 58^m.2$).
Lund, Astronomiska Observatorium, Katalog von 11800 Sternen der Zone $+35°$ bis $+40°$.
Paris, Observatoire, Catalogue d'étoiles fondamentales.
Mary Proctor, The Romance of Comets.
Pulkovo, Observatoire, Tables des quantités Besseliennes pour les années 1921, 1922–25, 1926–30.
W. F. Rigge, The Graphic Construction of Eclipses and Occultations.
San Fernando, Observatorio de Marina, Carta Fotográfica del Cielo, zona $-5°$ (10 charts).
Strasbourg, Observatoire, Annales, tome i. [New Series.]
Uccle, Observatoire Royal, Carte Photographique du Ciel, zones $+33°$, $+35°$, $+37°$, $+39°$ (20 charts).

Celestial Photographs.

The following have been added to the list of Celestial Photographs reproduced by the Society for sale to the Fellows :—

R.A.S.
Ref. No.
376. Spectroheliogram, Hα, Bright Filaments, 1926 Feb. 22, $8^h\ 8^m$, Kodaikanal.
377. Spectroheliogram, Hα, Bright Filaments, 1926 Feb. 22, $8^h\ 36^m$, Kodaikanal.
378. Spectroheliogram, K, Bright Filaments, 1926 Feb. 22, $9^h\ 0^m$, Kodaikanal.
379. Spectroheliogram, Hα, Bright Filaments, 1926 Feb. 22, $9^h\ 4^m$, Kodaikanal.
380. Neb. N.G.C. 5247 (Virgo), 1915 Mar. 21, 36-in. reflector, Lick.

R.A.S.
Ref. No.

381. Neb. N.G.C. 5921 (Serpens), 1912 June 10, 36-in. reflector, Lick.
382. Neb. N.G.C. 2903–5 (Leo), 100-in. reflector, Mount Wilson.
383. Neb. N.G.C. 4567–8 (Virgo), 60-in. reflector, Mount Wilson.
384. Neb. N.G.C. 5236 (M83. Hydra), 100-in. reflector, Mount Wilson.
385. Neb. N.G.C. 5850 (Libra), 100-in. reflector, Mount Wilson.
386. Neb. N.G.C. 7479 (Pegasus), 60-in. reflector, Mount Wilson.
387. Solar Corona, 1926 Jan. 14, Sumatra, 4-in. reflector, 19-ft. focus, Aston.
388. Spectrum of Chromosphere (1st contact), 1926 Jan. 14, Sumatra, 19-ft. prismatic camera, Aston.
389. Spectrum of Chromosphere (2nd contact), 1926 Jan. 14, Sumatra, 19-ft. prismatic camera, Aston.
390. Solar Eclipse Tracks, 1715, 1724, *Observed* Paths, Halley.

The late Dr. J. L. E. Dreyer has bequeathed to the Society the following books and photographs :—

Collection of Pamphlets on the History of Astronomy (3 vols.).

Struyck (Nicolaas) : Inleidung tot de Algemeene Geographie, benevens eenige Sterrekundige en andere Verhandelingen. 4to. Amsterdam, 1740.

———————— : Vervolg van de Beschryving der Staartsterren, ... benevens eenige Sterrekundige, Aardrykskundige en andere Aanmerkingen. 4to. Amsterdam, 1753.

Biot (Jean Baptiste) : Mélanges d'Astronomie (2 vols.). 4to. Paris. 1818–61.

Martin (Thomas Henri) : Recherches sur la vie et les ouvrages d'Héron d'Alexandre. 4to. Paris, 1854.

———————— : Astronomie ancienne. 4to. Paris and Rome, 1854–84.

———————— : Hypothèses Astronomiques Grecques. 4to. Paris and Rome, 1874–83.

Oeltzen (Wilhelm) : Astronomische Abhandlungen. 8vo. Vienna, 1854–56.

[This is a complete set of Oeltzen's papers on star-places and proper motions.]

Photographs of " Catalogus 723 Stellarum," " De Triangulis," and " Commentationes Davidis Fabricii." (Printed in Tycho Brahe, Opera.)

The Rotation of the Earth. By H. Spencer Jones, M.A., Sc.D., H.M. Astronomer at the Cape.

1. The slowly varying discrepancies which exist between the observed motion of the Moon and the theoretical motion derived by Newcomb, Brown, and others have given rise to considerable discussion. Newcomb gave much attention to this question, and amongst the possible explanations considered by him was the hypothesis that the deviation is only an apparent one, due to a slow variation in the rate of rotation of the Earth producing a corresponding change in the measure of time. After an exhaustive investigation of the long-period inequalities in the motion of the Moon, Newcomb interpreted the residual discordances between observation and theory as time-errors due to a variable rate of rotation of the Earth, and compared these with the time-errors derived from a discussion of the transits of Mercury.

Newcomb pointed out that

> " Les observations des passages de Mercure accusent nettement de petites variations dans la rotation de la Terre dont le montant intégré, pendant de longues périodes de temps, s' élève probablement à cinq, ou même à dix secondes. En particulier il semble que, entre 1769 et 1789, un ralentissement de la rotation avait lieu et que, entre 1840 et 1861, encore un autre. Vers 1862, ce ralentissement était suivi brusquement d'une accélération bien accentuée, qui a persisté peut-être jusqu'à 1870. Ce qui est remarquable, c'est que cette dernière conclusion est confirmée par le mouvement observé de la Lune." *

This correlation was merely a qualitative one, and it appeared that only about one-third of the Moon's residuals could be ascribed to variations in the rate of the Earth's rotation.† Newcomb therefore did not definitely accept the latter hypothesis as established, and his final conclusion was that

> " The evidence seems almost conclusive that the very improbable deviations in the Earth's rotation inferred from the observation of the Moon are unreal, and that the motion of our satellite is really affected by causes which have, up to the present time, eluded investigation." ‡

2. If, as Newcomb conjectured, the residual errors of the Moon's longitude are due in part to a variable rate of rotation of the Earth, the variation should be reflected in the motions of the Sun and the planets. The planets with the largest mean motions in longitude are the best adapted for the examination of this question, and the discussion may be confined to the Sun, Mercury, Venus, and Mars. A general correlation between the residuals of the Moon, the Sun, and Mercury was pointed out by Brown.§ The subject was investigated in more

* Newcomb, *Comptes Rendus*, t. cxxii., 1238, 1896.
† Newcomb, *Astron. Papers of the American Ephemeris*, **1**, 461-465.
‡ Newcomb, *M.N.*, *R.A.S.*, **63**, 318, 1903.
§ *Brit. Assoc. Report* (Australia, 1914): " Address on Cosmical Physics."

detail by Glauert,* who found a strong correlation for the period 1866 to 1914 between the residuals of the Moon, the Sun, Venus, and Mercury, the observational basis being provided by the Greenwich observations.

An observed error in geocentric longitude of a planet involves the errors in heliocentric longitude of the planet and of the Sun. Glauert assumed that both errors were due to changes in the rate of rotation of the Earth and that the heliocentric errors were in the ratio of the mean motions. On these assumptions, the errors of the Sun's longitude were derived separately from the observations of the Sun, Mercury, and Venus. As the Sun's error enters mainly into the geocentric error of Mercury and largely into the geocentric error of Venus, a general similarity between the resulting curves would be anticipated; the similarity found by Glauert is not therefore sufficient proof that the residuals in longitude of the Sun, Mercury, and Venus show a close correlation. Glauert's assumption that the errors are in the ratio of the mean motions, moreover, presupposes that the tabular longitudes at epoch and the mean motions are correct.

In a somewhat later paper, Glauert † met these objections to a certain extent by deriving the heliocentric longitude errors of Venus from the geocentric errors, using for the longitude errors of the Sun the values derived from observations of the Sun. The results in this case were much more irregular, though a general correlation between the residuals for Venus and the Sun was still apparent.

Observations of Venus made near inferior conjunction enter with predominant weight into the derivation of the error of heliocentric longitude. Now it is well known that observations of Venus and Mercury are subject to a marked discontinuity as the planet passes through inferior and superior conjunction, corresponding with the changes in the limb which is observed. It is not uncommon in the case of Venus for the discontinuity in the observed error of geocentric longitude, as the planet passes through inferior conjunction, to exceed three seconds of arc. Unless therefore the combined weights of the observations immediately preceding and following inferior conjunction are approximately equal, serious error may be introduced through neglecting to take this systematic error into account. Glauert's reductions are open to this objection.

The present paper gives the results of a re-examination of the information obtainable from observations of the Sun, Mercury, Venus, and Mars as to the variability of the rate of rotation of the Earth. The discussion was limited to observations made from 1836 onwards. It was not judged expedient to extend the investigation to an earlier period than this, on account of the large accidental errors of the earlier observations; it was not anticipated that the value of the results would be commensurate with the labour involved. In the case of Mercury the observations are relatively of small weight; it was therefore proposed to combine the results from the meridian observations with those derived from observations of transits. But before this part of

* *M.N., R.A.S.*, **75**, 489, 1915.
† *Ibid.*, **75**, 685, 1915.

the investigation was commenced, the rediscussion by Innes* of the transits from 1677–1924 appeared. The results derived by Innes have therefore been utilised for comparison with those deduced from the meridian observations.

It need hardly be mentioned that the Greenwich observations provide the principal material upon which the discussion is based.

3. *The Sun.*—The results of the discussion of the Greenwich observations of the Sun during the period 1836–1923, which was undertaken primarily in connection with the present investigation, have already been published.† For the period 1836–1863, the tabular places used in the Greenwich observations are from Carlini's tables. On account of the rather large errors in these tables, the reduction to the basis of Newcomb's tables was made by means of a direct comparison between Carlini's and Newcomb's tables for the mean date of observation in each calendar month as given in the Greenwich volumes. The weighted mean of the mean monthly differences between the tables and the observations, as thus corrected, was used to derive the mean longitude error for the year. To reduce to Newcomb's equinox, the corrections given in Table I. of the paper quoted were applied.

From 1864 to 1900 the tabular positions were derived from Le Verrier's tables. To reduce to Newcomb's tables, the differences in mean longitude, mean anomaly, and principal elliptic coefficient in longitude have to be allowed for. The expressions for the combined correction for the middle of each month are given on p. 428 of the previous paper. The corrections were applied in this form, and although the mean date in each month deviates on occasion by several days from the meridian date, the errors thus introduced will be practically inappreciable in the derivation of the mean error for the year.

From 1901 onwards the tabular positions are those of Newcomb's tables. The mean longitude error for each year is therefore directly obtainable from the annual volumes of Greenwich observations.

It is well known that observations of the Sun are peculiarly liable to systematic personal and instrumental errors. But the values of the errors of mean longitude of the Sun were required in the subsequent work for the derivation of the errors of heliocentric longitude of the planets from the observed geocentric errors, and it was therefore of considerable importance that the values to be adopted should be derived with as much accuracy as possible. For this reason the results derived from the Greenwich observations were combined with those resulting from observations made elsewhere. In a paper by Ross,‡ entitled "The Sun's Mean Longitude," are given the corrections to the mean longitude of the Sun from Newcomb's tables resulting from the several series of observations utilised by Newcomb in his *Astronomical Constants*, pp. 30–32, together with the following later series: Greenwich, 1893–1912; Paris, 1890–1906; Washington, 1894–1911. The results given by Ross for observatories other than Greenwich were utilised, with the addition of the Cape observations, 1907–16. The

* *Union Observatory Circ.*, No. 65, 1925.
† *M.N., R.A.S.*, **86**, 426, 1926.
‡ *Astronomical Journal*, **29**, No. 691, 152, 1916.

TABLE I.

Longitude Errors of Sun, 1836–90.

(Newcomb – Observed.)

Greenwich.	Paris.	Cambridge.	Washington.	Königsberg.	Oxford.	Pulkowa.	Dorpat.
1837–43 $-''\cdot32_1$	1837–44 $+''\cdot13_1$	1835–40 $+''\cdot40_1$		1839–45 $-''\cdot35_1$			1831–38 $-''\cdot32_1$
					1840–49 $-''2\cdot39_{0\cdot3}$	1842–50 $-''1\cdot15_1$	
1844–52 $-\cdot41_1$	1845–52 $-\cdot17_1$	1842–47 $+\cdot14_1$	1846–52 $+''\cdot74_1$				
1853–59 $-\cdot44_2$	1853–59 $-\cdot44_1$	1850–58 $+\cdot36_1$					
1860–65 $-\cdot20_2$	1860–65 $-\cdot59_1$		1861–65 $+\cdot27_2$				
					1860–68 $-2\cdot16_{0\cdot3}$	1861–70 $+\cdot20_1$	
1866–70 $-\cdot79_3$	1866–70 $-2\cdot42_{0\cdot3}$		1866–73 $+\cdot17_2$				
					1869–76 $-\cdot95_{0\cdot3}$		
1871–75 $-\cdot54_3$	1871–79 $+\cdot06_1$		1874–81 $+\cdot07_2$				
1876–80 $-\cdot29_3$							
						Cape.	Strasbourg.
1881–85 $-\cdot31_3$	1880–89 $-\cdot60_1$		1882–91 $+\cdot73_2$		1880–87 $+\cdot15_{0\cdot3}$	1884–90 $+\cdot26_3$	1883–88 $+1\cdot58_2$
1886–90 $-\cdot30_3$							

The weights given to the several series are indicated by the suffixes.

Greenwich values derived during the present investigation should be more accurate than those given by Ross, and were therefore used in preference to Ross's values, although the agreement is in general good.

The figures given by Ross bring out very clearly the systematic differences between the corrections to the tables resulting from the observations at different observatories.

Ross remarks that "the Washington observations of the Sun for sixty years have given a result $0^s{\cdot}05$ less in right ascension than Greenwich, and $0^s{\cdot}08$ less than Paris for the same period. On account of the great number of observers who must have taken part in the observations over such an extended period, it is inconceivable that this is a result of personal equation. It must accordingly be considered as of instrumental or housing origin or as a local systematic refraction."

In order to adopt as uniform a system as possible and to avoid discontinuities arising from the systematic differences between the several partial series utilised, the various series have been reduced to the Greenwich basis. This procedure is necessarily somewhat arbitrary, inasmuch as it is improbable that the systematic differences between several observatories will remain constant for several years at a stretch, during which period different observers have doubtless participated in the observations. Nevertheless, judging from the comparison of the longest series of observations, those secured at Greenwich, Washington, and Paris, the greater portion of the differences between the several series will be eliminated in this way, and the combined mean should eliminate to a large extent the accidental errors of observation inherent in any single series.

For the period 1836–1890, the available material is summarised in Table I.

To the Cambridge observations a systematic correction of $-0''{\cdot}69$ was applied, and to the Washington observations a correction of $-0''{\cdot}78$, in order to reduce to the basis of Greenwich. The following values of the longitude errors given in Table II. were adopted as best representing the several series.

TABLE II.

Adopted Values of Sun's Longitude Errors, 1836–90.

1839	$-0''{\cdot}16$	1873	$-0''{\cdot}51$
47·5	$-\ {\cdot}29$	78	$-\ {\cdot}35$
55·5	$-\ {\cdot}40$	83	$-\ {\cdot}29$
63	$-\ {\cdot}43$	88	$-\ {\cdot}28$
69	$-\ {\cdot}82$	92·5	$-\ {\cdot}01$

For the period from 1890 onwards, the errors resulting from the observations at Greenwich, Paris, Washington, and the Cape are given year by year in Table III.

TABLE III.

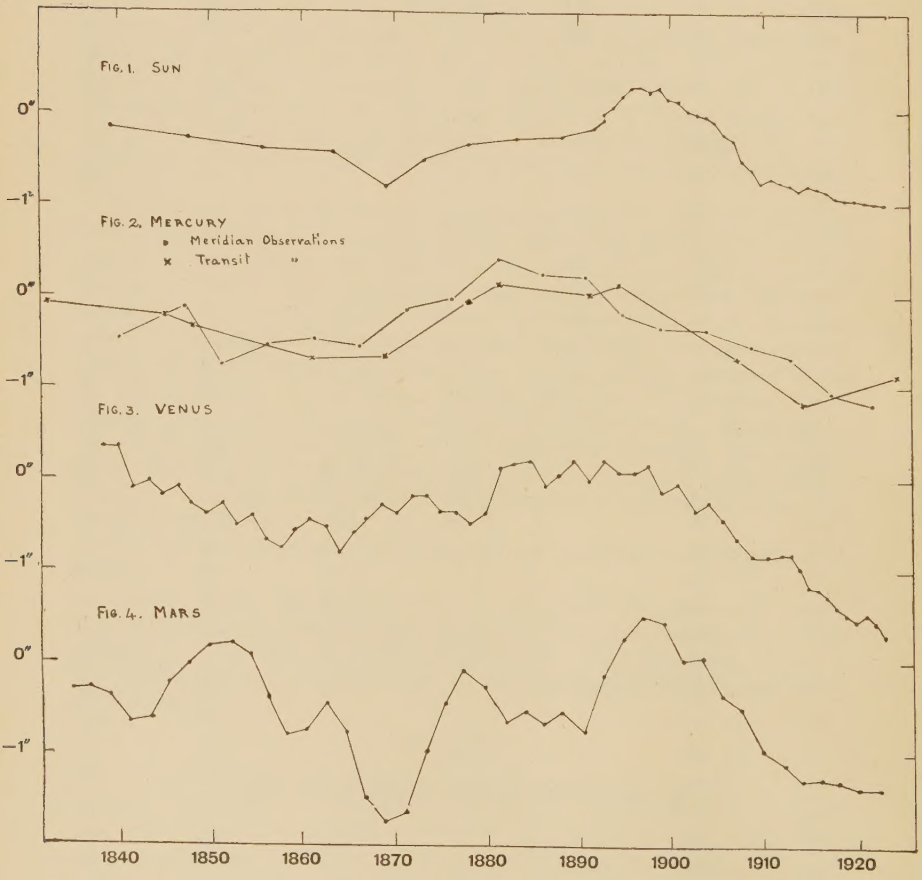
Longitude Errors of Sun, 1890–1923.

Year.	Greenwich.	Paris.	Washington.	Cape.	Adopted Mean.	Smoothed Means.
	″	″	″	″	″	″
1890	−0·48	−0·82	..	..	−0·41	
91	− ·06	− ·66	..	..	− ·06	−0·17
92	+ ·03	− ·85	..	..	− ·04	− ·07
93	− ·06	− ·88	..	..	− ·11	+ ·07
94	+ ·36	..	..	..	+ ·36	+ ·18
95	+ ·29	..	..	..	+ ·29	+ ·27
96	+ ·22	..	+0·64	..	+ ·15	+ ·27
97	+ ·40	− ·12	+ ·82	..	+ ·37	+ ·21
98	+ ·19	− ·46	+ ·51	..	+ ·11	+ ·26
99	+ ·26	− ·18	+ ·96	..	+ ·30	+ ·15
1900	+ ·04	− ·64	+ ·78	..	+ ·04	+ ·11
01	+ ·05	− ·36	+ ·25	..	− ·01	+ ·01
02	+ ·10	− ·72	+ ·54	..	+ ·01	− ·01
03	+ ·11	− ·83	+ ·48	..	− ·02	− ·05
04	− ·27	− ·23	+ ·48	..	− ·13	− ·10
05	− ·25	− ·88	+ ·90	..	− ·16	− ·23
06	− ·31	−1·34	+ ·29	..	− ·41	− ·30
07	− ·31	..	+ ·58	−1·32	− ·33	− ·52
08	− ·96	..	+ ·24	−1·53	− ·83	− ·62
09	− ·60	..	− ·04	−1·68	− ·70	− ·78
1910	− ·84	..	+ ·19	−1·80	− ·82	− ·72
11	− ·92	..	+ ·33	− ·80	− ·64	− ·77
12	− ·85	..	..	−1·16	− ·84	− ·80
13	− ·88	..	..	−1·35	− ·91	− ·85
14	− ·86	..	..	− ·96	− ·80	− ·80
15	− ·67	..	..	−1·10	− ·69	− ·82
16	− ·94	..	..	−1·44	− ·98	− ·86
17	− ·90	..	..	..	− ·90	− ·93
18	− ·92	..	..	..	− ·92	− ·96
19	−1·05	..	..	..	−1·05	− ·96
1920	− ·90	..	..	..	− ·90	− ·98
21	− ·98	..	..	..	− ·98	− ·99
22	−1·08	..	..	..	−1·08	−1·00
23	− ·93	..	..	..	− ·93	

The Greenwich results from 1915 onwards have been corrected for the marked difference of personality of $0^s{\cdot}040$ which resulted from the introduction of the travelling-wire micrometer.* The assumption has necessarily had to be made that the mean personality of the Greenwich observers previously to 1915 remained constant. It is the only assumption which it is possible to make under the circumstances.

* Cullen, *M.N.*, *R.A.S.*, **85**, 1003, 1923.

In deriving the figures given in column headed "Adopted Mean," the following mean corrections have been applied to the Paris, Washington, and Cape results to reduce them to the basis of the Greenwich figures: to Paris, +0″·62; to Washington, −0″·69; to the Cape series, 1907–11, +0″·70; to the Cape series, 1912–16, +0″·36. The means have been derived by giving a weight of 3 to Greenwich and 1 to each of



the other observatories. In order to smooth out to some extent the accidental irregularities of the final yearly means, the means by threes are given in the final column of the table.

The longitude errors derived in Table II. and in the last column of Table III. have been plotted in fig. 1, and from a smooth curve drawn through the plotted points the values given in Table IV. have been derived. These values have been adopted as the definitive values for the error of the Sun's mean longitude as given by Newcomb's tables,

and have been used in the reduction of the observations of Mercury, Venus, and the Sun.

TABLE IV.

Final Adopted Errors of Sun's Longitude, 1836–1924.

(Newcomb–Observed.)

	1830.	1840.	1850.	1860.	1870.	1880.	1890.	1900.	1910.	1920.
+0	..	−0·17	−0·32	−0·43	−0·80	−0·32	−0·24	+0·10	−0·76	−0·97
1	..	− ·19	− ·34	− ·44	− ·73	− ·31	− ·17	+ ·02	− ·80	− ·99
2	..	− ·20	− ·35	− ·47	− ·62	− ·30	− ·08	− ·02	− ·82	−1·00
3	..	− ·22	− ·36	− ·51	− ·53	− ·30	+ ·05	− ·07	− ·84	−1·02
4	..	− ·23	− ·38	− ·57	− ·48	− ·30	+ ·18	− ·13	− ·86	
5	..	− ·25	− ·39	− ·63	− ·44	− ·31	+ ·25	− ·20	− ·88	
6	−0·12	− ·26	− ·40	− ·68	− ·40	− ·31	+ ·27	− ·37	− ·90	
7	− ·13	− ·28	− ·41	− ·74	− ·38	− ·30	+ ·24	− ·52	− ·91	
8	− ·15	− ·29	− ·42	− ·79	− ·36	− ·29	+ ·20	− ·64	− ·93	
9	− ·16	− ·31	− ·43	− ·82	− ·34	− ·27	+ ·15	− ·73	− ·95	

The curve in fig. 1 representing the errors shows a gradual downward trend from the beginning of the period under consideration until about 1870, followed by a steady, though not uniform, rise until 1896, after which year a rapid progressive fall has ensued.

From the detailed discussion of the Greenwich observations of the Sun during this period, a correction to Newcomb's value of the mean longitude of amount +0″·47+0″·34T was derived. This is the correction which brings observation and theory into closest agreement during this period. Fig. 1 shows, however, that corrections to Newcomb's longitude at epoch and mean motion do not suffice to represent the observations within the limits of accidental error, and that there are residual systematic discordances which are not accounted for by theory.

4. *Mercury.*—Meridian observations of Mercury are not of very great value for the purpose of the present discussion. The observations are relatively scanty in number and in general not of very great weight for the determination of the error of heliocentric mean longitude of the planet. Observations close to inferior conjunction are of the greatest value for this purpose, but cannot be secured in the case of Mercury, as the planet is then too close to the Sun to be observable. Observations of the times of the contacts during transits of Mercury over the Sun provide results of much greater accuracy. Nevertheless, it is of some interest to examine to what extent the meridian observations confirm the results derived from the observations of transits. Considerable trouble has been taken in the derivation of the results, and it is satisfactory to find that there is a good general agreement between the results furnished by the two methods.

The discussion has been limited to the Greenwich observations and to the few series of observations secured at the Cape. Owing to the large amount of computation involved in the reductions, it did not appear

worth while to extend the discussion to observations secured elsewhere, particularly as, even if this had been done, the meridian results could not exceed in precision those derived from the observations of transits.

The tabular places of Mercury given in the *Nautical Almanac* were based upon Lindenau's tables from 1836 to 1863; upon Le Verrier's tables from 1864 to 1900, and from 1901 onwards they have been based upon Newcomb's tables. In the volumes of Greenwich observations, the observed place has been compared throughout with the *Nautical Almanac* place. In order to reduce to the basis of Newcomb's tables, direct comparison was made for noon of the date of each group mean given in the Greenwich observations for the period 1836 to 1900 between the tabular heliocentric longitude and radius vector derived from Newcomb's tables and the corresponding values as given in the *Nautical Almanac*. For the period 1864 to 1900, during which the *Nautical Almanac* places were derived from Le Verrier's tables, it was not considered sufficient merely to correct for the difference between Le Verrier's and Newcomb's mean longitudes, on account of the probability that, due to the relatively small yearly number of observations of Mercury at Greenwich, the differences in the short-period terms in the two tables would not thereby be sufficiently eliminated.

It has already been mentioned that observations of the planets Mercury and Venus, taken before and after inferior conjunction respectively, differ systematically. There is, therefore, a difference in the longitude corrections derived from observations of the two limbs, such as might result from the reduction of the observations with an incorrect value of the semi-diameter. The phenomenon is doubtless an irradiation or contrast effect, and its magnitude is greater at inferior than at superior conjunction. It might be anticipated that for a given instrument and magnification the magnitude of the phenomenon would be constant, apart from variations in personality of the observers, but that the phenomenon would differ considerably from one instrument to another of different aperture and magnification. This appears to be the case.

To eliminate this systematic error, it is best to collect the derived geocentric longitude corrections according to the angle of elongation, $l-\mathrm{L}$, where l, L denote the heliocentric longitudes of the planet and the Earth respectively. The theoretical expression for the geocentric longitude correction, $\Delta\lambda$, in terms of the corrections to the heliocentric longitudes Δl, $\Delta\mathrm{L}$, is

$$\Delta\lambda = \Delta a_0+\Delta\mathrm{L}+a(\Delta\mathrm{L}-\Delta l)\cos(l-\mathrm{L})\{1+a\cos 2(l-\mathrm{L})+\ldots\},$$

where Δa_0 denotes the equinox correction and a denotes the ratio of the mean distances of planet and Earth.

Hough * and Jackson,† in discussing the Cape and Greenwich observations, respectively analysed the longitude corrections in the form

$$a+b\cos(l-\mathrm{L})+c\sin(l-\mathrm{L})$$

and found an appreciable coefficient for the $\sin(l-\mathrm{L})$ term. This term, which is without dynamical meaning, arises from the systematic

* *Cape Annals*, **8**, pt. iv., p. 25D. † *M.N., R.A.S.*, **84**, 479, 1924.

error referred to above. To proceed in this way is therefore to assume that the error in question varies according to the sine of the elongation angle. This is not the case.

A better method is to group the geocentric longitude corrections according to $(l-\mathrm{L})$ and to take differences for corresponding values θ and $2\pi-\theta$. The values of $\Delta\lambda$ for these two angles should be equal, and the differences are due partly to accidental errors of observation and partly to the systematic error under discussion. The differences thus derived are plotted against $(l-\mathrm{L})$ and a smooth curve drawn to pass through the plotted points as nearly as possible, due regard being paid to the weights of the different points. From this curve, the values for different values of $(l-\mathrm{L})$ are tabulated, and a correction of half this amount is subtracted from the longitude correction for elongation θ and added to the longitude correction for elongation $2\pi-\theta$. This procedure in effect assumes that the mean correction derived from a series of observations symmetrical in number and in position in the relative orbit, before and after inferior conjunction, would be free from systematic error. This is the most rational assumption which can be made under the circumstances.

The Greenwich observations of Mercury were discussed in groups of about twelve years ; the geocentric longitude corrections were arranged according to the angle $(l-\mathrm{L})$, and mean corrections for the period derived. From the smoothed curve, corrections to the individual observations were read off and applied. In the volumes of Greenwich observations up to and including the year 1914, the geocentric longitude errors are expressed in terms of the errors of heliocentric longitudes and radii vectors of the planet (l, r) and the Earth (L, ρ) in the form

$$\Delta\lambda = \alpha\Delta l+\beta\Delta r+\gamma\Delta\mathrm{L}+\delta\Delta\rho,$$

where $\gamma = (1-\alpha)$ approximately. The radii vectors of Newcomb's tables were assumed correct, and corrections Δr, $\Delta\rho$, resulting from the comparison of the radii computed from Newcomb's tables with the values given in the *Nautical Almanac* and computed from the tables of Lindenau, Carlini, or Le Verrier, were applied up to the year 1901. After 1901, the corrections Δr, $\Delta\rho$ were neglected. Corrections for the longitude differences between Newcomb's and the other tables used in the *Nautical Almanac,* and equinox corrections to reduce to Newcomb's equinox, were also applied, the equations thereby being reduced to the form

$$\alpha\Delta l+(1-\alpha)\Delta\mathrm{L} = c,$$

where c denotes the constant term which results after the corrections have been applied for the systematic difference depending upon the limb observed, for the differences in heliocentric longitudes and radii vectors between Newcomb's and the other tables, and for equinox. Δl, $\Delta\mathrm{L}$ now denote differences between observation (referred to Newcomb's equinox) and Newcomb's tables.

From the equations so corrected, the normal equation for the error of the mean longitude of the planet was formed. This equation involves $\Delta\mathrm{L}$, the error of the mean longitude of the Earth : this is derived

with only small weight from the planetary observations. The normal equation for ΔL was therefore not computed, but the errors derived from solar observations, and given in Table IV., were substituted in the normal equation for Δl and the error of the mean longitude of Mercury derived.

The observations of Mercury are not sufficiently numerous to render it possible to discuss the observations year by year. They were therefore grouped a few years at a time so as to give fairly equal weights to each group.

After 1914 the errors of geocentric longitude resulting from the Greenwich observations are not directly expressed in terms of the heliocentric errors. For this period, from the errors in right ascension and declination the errors in heliocentric longitude were computed. These were grouped according to $(l-L)$, and after correction for the systematic error depending upon the limb observed, were analysed in the form $\alpha+\beta\cos(l-L)$. The coefficient β is equal to $a(\Delta L-\Delta l)$, a being the ratio of the mean distances, viz. 0·387. From the derived value of β, therefore, by substituting the value of ΔL from Table IV., the correction to the planet's mean longitude is obtained.

The Cape series of observations, 1884–1892 and 1907–1916, were discussed in a similar manner to the Greenwich observations from 1914 onwards.

The results derived for this discussion are given in Table V. The

TABLE V.

Errors of Mean Longitude of Mercury, 1836–1923.

(Newcomb−Observed.)

Period.	Greenwich.	Cape.	Adopted Δl.	$\cdot 241 \times \Delta l$.
	″	″	″	″
1836–43	−1·99	..	−1·99	−0·48
44–48	−0·47	..	−0·47	−0·11
49–53	−3·29	..	−3·29	−0·79
54–58	−2·16	..	−2·16	−0·52
59–63	−1·94	..	−1·94	−0·47
64–68	−2·36	..	−2·36	−0·57
69–73	−0·50	..	−0·50	−0·12
74–78	−0·13	..	−0·13	−0·03
79–83	+1·74	..	+1·74	+0·42
84–88	+0·60	+1·16	+0·97	+0·23
89–92	−0·65	+1·61	+0·86	+0·21
93–96	−0·70	..	−0·70	−0·17
1897–00	−1·28	..	−1·28	−0·31
1901–06	−1·42	..	−1·42	−0·34
07–10	−2·89	−1·86	−2·20	−0·53
11–14	−3·48	−2·38	−2·75	−0·66
15–19	−4·26	−4·16 *	−4·22	−1·02
20–23	−4·76	..	−4·76	−1·15

The errors contained in the last column are plotted in fig. 2 (p. 10).

* 1915, 1916 only.

first column gives the period covered by the group, the second and third columns give the errors of heliocentric longitude of Mercury derived from Greenwich and Cape observations respectively in the sense Newcomb's tables *minus* Observed; the fourth column gives the adopted correction, the Cape observations being given double the weight of those at Greenwich, on account of their greater number. The last column gives the error multiplied by 0·241—the ratio of the mean motions of the Earth and Mercury. If the residual errors of longitude are attributable to the variation of the Earth's rotation, they will for different bodies be proportional to their mean motions. It is therefore the quantities in the last column which are to be compared with the errors of the Earth's longitude derived from observations of the Sun.

5. *Mercury: Observations of Transits.*—The observations of transits of Mercury from 1677 to 1924 have recently been discussed with great thoroughness by Innes.* In this discussion the tabular times of contact have been computed throughout on the basis of Newcomb's tables, and the results are given (p. 312) as time-error resulting from variability of the rotation of the Earth. These time-errors may be interpreted as arising from the use of incorrect tabular positions of the Sun and Mercury, and the necessary data for the derivation of the resulting errors are given in the paper. For, with Innes's notation, if l, b; l', b' denote the heliocentric longitudes and latitudes of Mercury and the Earth respectively, M denotes the position angle of Mercury from the Earth, and N is a quantity derived from the hourly motions by the relationships

$$l'-l = n \sin \mathrm{N}$$
$$b-b' = n \cos \mathrm{N},$$

then the time-error $\Delta\tau$ is given in terms of the errors of the longitudes by the relationship

$$\Delta\tau n \cos (\mathrm{M}-\mathrm{N}) = (\Delta l'-\Delta l) \sin \mathrm{M}.$$

$\Delta\tau$ and $-\sin \mathrm{M}/n \cos (\mathrm{M}-\mathrm{N})$ are tabulated in Table VIII., p. 320, of Innes's paper, and therefore $\Delta l-\Delta l'$ may be inferred. Using the figures for the Sun's errors given in Table IV., the error of mean longitude of Mercury is derived. The resulting figures, being the weighted mean values derived from each transit, are given in Table VI., together with the same quantities multiplied by the ratio of the mean motions of the Earth and Mercury.

The probable error of the concluded time of a transit may be taken as about $\pm 2^{\mathrm{s}}$. The coefficient for deriving $\Delta(l-l')$ from $\Delta\tau$ ranges from 4 to 15, so that the probable error of $\Delta(l-l')$ should not exceed $\pm 0''{\cdot}05$ and the probable error of $0{\cdot}241 \times \Delta l$, neglecting any uncertainty resulting from the adopted errors of longitude of the Sun, will only be of the order $\pm 0''{\cdot}01$. This greatly exceeds the accuracy of which meridian results are capable. The reduced errors contained in the last column of Table VI. have been plotted in fig. 2 along with those resulting

* *Union Observatory Circ.*, No. 65, 1925.

from the meridian observations. It is satisfactory to find that, in spite of the relatively small weights of the results from the meridian observations, there is a close general agreement between the two curves, the rise from about 1868 and the steady fall in the later years being prominent in each.

TABLE VI.

Errors of Mean Longitude of Mercury from Transits.

(Newcomb−Observed.)

Date.	Δl.	$\cdot 241 \times \Delta l$.
	″	″
1832 May 4–5	−0·39	−0·09
45 ,, 8	−0·87	−0·21
48 Nov. 8–9	−1·44	−0·35
61 ,, 11	−2·88	−0·69
68 ,, 4	−2·80	−0·67
78 May 6	−0·26	−0·06
81 Nov. 7	+0·94	+0·23
91 May 9	+0·09	+0·02
94 Nov. 10	+0·56	+0·13
1907 ,, 13–14	−2·78	−0·67
14 ,, 6–7	−4·73	−1·14
24 May 7	−3·50	−0·84

6. *Venus: Meridian Observations.*—The heliocentric longitude of Venus can be determined with greater precision than that of Mercury, on account both of the larger number of observations available and of the relatively closer proximity to inferior conjunction at which observations can be secured. Meridian observations of Venus are therefore of greater value for the present discussion than those of Mercury. The Greenwich observations in themselves provide a very smooth determination of the errors of the planet's mean longitude, and, as in the case of Mercury, the discussion has been limited to the Greenwich series supplemented by the relatively few Cape observations.

The method of treating the observations was generally similar to that adopted in the case of the observations of Mercury. The tabular places of Venus given in the *Nautical Almanac* were based upon Lindenau's tables from 1836 to 1864, upon Le Verrier's tables from 1865 to 1900, and from 1901 onwards they have been based upon Newcomb's tables. In the Greenwich volumes of observations, comparison is made throughout between the observed places and the tabular places of the *Nautical Almanac*. For the period 1836 to 1864 the tabular places were reduced to the basis of Newcomb's tables by comparing directly the *Nautical Almanac* heliocentric longitude with the longitude computed from Newcomb's tables. The comparison was made for noon of the date of each group mean given in the Greenwich observations. The errors of the radius vector of Lindenau's tables of Venus being large, the difference in log radius vector between Lindenau's and Newcomb's tables was also computed.

For the period during which Le Verrier's tables were used in the *Nautical Almanac*, it was considered sufficient to allow for the differences in mean longitude and in long-period terms between Le Verrier's and Newcomb's tables. The observations are sufficiently numerous for differences in short-period terms to be negligible in the mean.

The mean longitude of Venus according to Newcomb is

$$344^\circ\ 22'\ 9''{\cdot}20+210669162''{\cdot}88\mathrm{T}+1''{\cdot}1148\mathrm{T}^2,$$

and according to Le Verrier is

$$344^\circ\ 22'\ 11''{\cdot}05+210669166''{\cdot}17\mathrm{T}+1''{\cdot}1289\mathrm{T}^2,$$

where T is the number of centuries of 36,525 days measured from 1900 January 1 mean noon.

Hence we have for the difference in mean longitude, Le Verrier *minus* Newcomb:

$$\Delta l = +1''{\cdot}85+3''{\cdot}29\mathrm{T}+0''{\cdot}014\mathrm{T}^2.$$

Of the long-period terms in the longitude, the term depending upon the product of the masses of Mars and the Earth and that depending on the mean anomaly of Saturn, as given by Newcomb and Le Verrier respectively, are in essential agreement. There is a difference between the two tables, however, in the long-period terms, due to the action of the Earth.

Newcomb gives for this term

$$(2''{\cdot}761-0''{\cdot}022\mathrm{T})\sin(13g'-8g+223^\circ\ 23'+271'\mathrm{T}),$$

and Le Verrier gives

$$+1''{\cdot}717\sin(13l''-8l')-1''{\cdot}905\cos(13l''-8l').$$

The difference, in the sense Le Verrier *minus* Newcomb, is approximately

$$-0''{\cdot}20\sin(237^\circ{\cdot}24+150^\circ{\cdot}27\mathrm{T}).$$

For the reduction from Le Verrier to Newcomb the mean longitude difference has therefore been taken as

$$+1''{\cdot}85+3''{\cdot}29\mathrm{T}+0''{\cdot}014\mathrm{T}^2-0''{\cdot}20\sin(237^\circ{\cdot}24+150^\circ{\cdot}27\mathrm{T}).$$

As in the case of Mercury, the systematic differences between observations secured before and after inferior conjunction were allowed for by forming the mean longitude errors for different values of $l-\mathrm{L}$ for periods of about ten years. The derived differences in mean longitude error for elongations θ and $2\pi-\theta$ were plotted against θ and a smooth curve drawn through the plotted points from which the corrections were read off to reduce all observations to the mean.

The equations in the Greenwich volume expressing the geocentric error of Venus in terms of the errors of heliocentric longitude and radius vector of both Sun and Earth were used: corrections were applied for the differences in longitude and radius vector of Venus between Newcomb's and Lindenau's tables, for the differences in longitude and radius vector of the Sun between Newcomb's and Carlini's tables (interpolated from the differences derived in discussing the solar observa-

tions), for difference between the equinox of the Greenwich observations and Newcomb's equinox, and for the systematic difference depending upon (l–L) referred to above. Finally, an equation of the form

$$\alpha\Delta l+(1-\alpha)\Delta L = c$$

is obtained, Δl, ΔL denoting the differences between observation and Newcomb's tables.

As the coefficients α are largest near inferior conjunction, the observations near the time of inferior conjunction have the greatest weight. The synodic period of Venus being 584 days, the weight of the final determination of Δl from a year's observations depends to a large extent upon whether or not an inferior conjunction occurs during the year. It is therefore more convenient to combine all the observations secured from one superior conjunction to the next. For this interval the normal equation for Δl was formed, and the value of ΔL derived from the observations of the Sun was used to deduce the value of Δl.

For the period 1869–1900 the discussion of the Greenwich observations of Venus by Cowell* was utilised. Cowell tabulates the mean error of right ascension of Venus at intervals of 15° of V–E, interpolating values where necessary. He expresses the error of geocentric longitude, $\Delta\lambda$, in the form

$$\Delta\lambda = ax+a'x'+by+b'y'+cz+c'z',$$

where $x+x'$, x' denote the errors of the mean longitude of Venus and the Earth respectively, and the other terms depend on the corrections required by the perihelion and eccentricity of Le Verrier's tables. These coefficients are such that, when equal weight is given to each $\Delta\lambda$, the mean values of aa', ab, ab' . . . are all zero, and x, the difference between the errors of mean longitude of Venus and the Earth, is then derived from the normal equation:

$$[aa]x = [a\,.\,\Delta\lambda].$$

The values of a are given in Cowell's Table I.†

Cowell's method has the great advantage of introducing symmetry into the observations, also since observations secured at values θ and $2\pi-\theta$ of V–E receive the same weight, the coefficient a being the same for these two angles, the systematic difference between observations secured before and after inferior conjunction is automatically eliminated. Some sacrifice in the weight of the derived longitude correction is necessarily involved. In Cowell's discussion the data from the Greenwich observations are tabulated in a form convenient for use in the present investigation, and to avoid the considerable labour involved in a discussion *ab initio* for the period 1869–1900 similar to that of the preceding years, the figures given by Cowell have been utilised. From them the values of x, or $\Delta l-\Delta$L in our previous notation, were derived for each period from one superior conjunction to the next, and obtained by using the errors of the Sun's longitude given in Table IV. The Cape series of observations during the period 1884–1892 were discussed in the same way.

* *M.N., R.A.S.*, **66**, 307, 1906.

† *Loc. cit.*, p. 313.

From 1901 Newcomb's tables have been used for the *Nautical Almanac* positions both of Venus and the Sun. The equations in the Greenwich volumes for these years connecting the geocentric longitude error of Venus with the errors of heliocentric longitude and radius vector of Venus and the Earth were used, but the errors of the radii vectores were neglected. The corrections for the systematic error depending upon the limb observed were determined and applied. The observations were discussed from 1901 onwards, year by year.

Since 1914, the geocentric longitude errors and their expression in terms of the heliocentric longitude errors have not been given in the Greenwich volumes. For these years the errors of geocentric longitude were computed from the errors of right ascension and declination, and collected according to the angle $(l-\mathrm{L})$. After correction for the systematic error depending upon the limb observed, these were analysed in the form $\alpha+\beta\cos(l-\mathrm{L})$. The coefficient β is equal to $a(\Delta\mathrm{L}-\Delta l)$, where a, the ratio of the mean distances, has the value 0·723. Having determined β, by substituting the value of $\Delta\mathrm{L}$ from Table IV., the correction to the mean longitude of Venus is derived.

The Cape series of observations, 1907–1916, were discussed in a manner similar to that of the Greenwich observations from 1914 onwards.

The results are summarised in Table VII., which gives the derived errors of mean longitude of Venus in the sense Newcomb *minus* Observed. The first column gives the date, the second column the error derived from the Greenwich observations. The third column gives the error derived from the Cape series, and the fourth column the mean of the errors according to Greenwich and the Cape for those years in which Cape observations are available. The fifth column gives the smoothed results obtained by taking the means of three years at a time. The final column gives the longitude errors multiplied by 0·615, the ratio of the mean motions of the Earth and Venus. If the residual errors are due to variations in the rate of rotation of the Earth, the figures in the latter column should be comparable with those obtained from observations of the Sun and Mercury.

TABLE VII.

Errors of Mean Longitude of Venus, 1836–1923.

(Newcomb–Observed.)

Date.	Δl (Gwh.).	Δl (Cape).	Mean.	Δl smoothed.	$\cdot 615 \times \Delta l$.
	″	″	″	″	″
1836·6	+0·46				
38·2	+1·00	..	..	+0·56	+0·34
39·8	+ ·23	..	..	+ ·53	+ ·33
1841·4	+ ·35	..	..	− ·15	− ·09
43·0	−1·03	..	..	− ·02	− ·01
44·6	+ ·61	..	..	− ·30	− ·18
46·2	− ·47	..	..	− ·15	− ·09
47·8	− ·59	..	..	− ·40	− ·25

TABLE VII.—*continued.*

Date.	Δl (Gwh.).	Δl (Cape).	Mean.	Δl smoothed.	$\cdot 615 \times \Delta l$.
	″	″	″	″	″
1849·4	− ·13	..	..	− ·61	− ·38
1851·0	−1·12	..	..	− ·43	− ·26
52·6	− ·05	..	..	− ·82	− ·50
54·2	−1·28	..	..	− ·64	− ·39
55·8	− ·60	..	..	−1·05	− ·65
57·4	−1·28	..	..	−1·22	− ·75
59·0	−1·77	..	..	− ·90	− ·55
1860·6	+ ·36	..	..	− ·70	− ·43
62·1	− ·70	..	..	− ·83	− ·51
63·7	−2·16	..	..	−1·30	− ·80
65·3	−1·05	..	..	− ·93	− ·57
66·9	+ ·43	..	..	− ·69	− ·42
68·5	−1·45	..	..	− ·44	− ·27
1870·1	− ·29	..	..	− ·58	− ·36
71·7	·00	..	..	− ·28	− ·17
73·3	− ·54	..	..	− ·28	− ·17
74·9	− ·30	..	..	− ·54	− ·33
76·5	− ·77	..	..	− ·55	− ·34
78·1	− ·58	..	..	− ·76	− ·47
79·7	− ·92	..	..	− ·62	− ·38
1881·3	− ·37	..	..	+ ·19	+ ·12
82·9	+1·86	..	..	+ ·27	+ ·17
84·5	−1·05	−0·30	−0·67	+ ·32	+ ·20
86·1	− ·42	− ·02	− ·22	− ·13	− ·08
87·7	+ ·77	+ ·22	+ ·49	+ ·05	+ ·03
89·3	− ·62	+ ·39	− ·11	+ ·33	+ ·20
1890·9	+1·51	− ·30	+ ·60	·00	·00
92·5	−1·07	+ ·10	− ·48	+ ·32	+ ·20
94·1	+ ·84	..	..	+ ·15	+ ·09
95·7	+ ·10	..	..	+ ·14	+ ·09
97·3	− ·51	..	..	+ ·22	+ ·14
98·9	+1·08	..	..	− ·19	− ·12
1900·5	−1·13	..	..	− ·08	− ·05
02·1	− ·18	..	..	− ·52	− ·32
03·7	− ·23	..	..	− ·38	− ·23
05·3	− ·72	..	..	− ·69	− ·42
06·9	−1·10	−1·12	−1·11	−1·03	− ·63
08·5	− ·93	−1·62	−1·27	−1·33	− ·82
1910·1	−1·32	−1·91	−1·61	−1·35	− ·83

TABLE VII.—*continued.*

Date.	Δl (Gwh.).	Δl (Cape).	Mean.	Δl smoothed.	$\cdot 615 \times \Delta l$.
	″	″	″	″	″
1911·7	−1·30	−1·06	−1·18	−1·30	− ·80
12·5	− ·78	−1·46	−1·12	−1·33	− ·82
13·5	− ·98	−2·43	−1·70	−1·56	− ·96
14·5	−2·10	−1·64	−1·87	−1·83	−1·13
15·5	−2·89	− ·93	−1·91	−1·88	−1·16
16·5	−1·41	−2·29	−1·85	−2·03	−1·25
17·5	−2·33	..	..	−2·21	−1·36
18·5	−2·46	..	..	−2·34	−1·44
19·5	−2·23	..	..	−2·46	−1·51
1920·5	−2·68	..	..	−2·34	−1·44
21·5	−2·11	..	..	−2·46	−1·51
22·5	−2·59	..	..	−2·74	−1·69
23·5	−3·53				

The errors contained in the last column have been plotted in fig. 3. There is a marked general resemblance to the run of the residuals of the Sun and Mercury as shown in figs. 1 and 2. In particular, the decline to about 1865, the marked rise from there amounting to about 1″, and the steady fall from about 1895 may be noted.

7. *Observations of Mars.*—The rediscussion by Ross * of observations of Mars has made it possible to supplement the present results by including the data relating to Mars. Ross first derives the orbital elements and mean motions from all available observations, and then substitutes these elements and derives the longitude residuals on two hypotheses: (1) that there are fluctuations in the longitude of Mars only; † (2) that there are fluctuations in the longitude both of the Earth and Mars.‡

The second solution has little weight, as the observations of Mars have in general been secured near opposition, so that there is a strong correlation between the coefficients of the longitude residuals for the Earth and Mars. It would have been better to substitute the values of the longitude residuals of the Sun as derived from solar observations, and then to derive the longitude residuals for the planet with unreduced weight. Ross did not do this, but it is possible to work back and proceed in this manner, since Ross gives the weights of the longitude residuals derived for each opposition from the two solutions.

As far as the heliocentric longitude corrections are concerned, the equation of condition resulting from any observation is

$$\Delta\lambda = \frac{r}{\Delta}\cos(\lambda - l)\Delta l - \frac{\mathrm{R}}{\Delta}\cos(\lambda - \mathrm{L})\Delta \mathrm{L},$$

* "New Elements of Mars," *Astron. Papers of the American Ephemeris*, 9 pt. ii., 1917.

† *Ibid.*, Solution (*a*). Table G, p. 266.

‡ *Ibid.*, Solution (*b*). Table H, p. 267.

where
R = Earth's radius vector.
r = Mars' radius vector.
Δ = distance, planet—Earth.
λ = geocentric longitude of Mars.
l = heliocentric longitude of Mars.
L = heliocentric longitude of Earth.

Or

$$\Delta\lambda = \kappa\Delta l - (\kappa - l)\Delta L,$$

where κ has a value about 3·0 at opposition and 1·25 at quadrature.

If we write $\bar{\kappa} = x$, $\bar{\kappa^2} = y$, the normal equations for Δl, ΔL are of the form

$$y \,.\, \Delta l - (y-x)\Delta L = \ldots$$
$$-(y-x)\Delta l + (y-2x+1)\Delta L = \ldots .$$

Then, in Ross's solution (*b*), the weights are :

of Δl $$y - \frac{(y-x)^2}{y-2x+1} = \frac{y-x^2}{y-2x+1} = w_1 \text{ (say)}$$

of ΔL $$y-2x+1-\frac{(y-x)^2}{y} = \frac{y-x^2}{y} = w_2 \text{ (say)}.$$

At opposition $y = x^2$, and hence if the observations are confined to opposition there is no weight for Δl, ΔL separately.

If Δl alone is determined, ΔL being neglected, the weight of the solution for Δl (Ross's solution (*a*)) is

$$y = w_3 \text{ (say)}.$$

Now this solution really determines $\Delta l - \frac{(y-x)}{y}\Delta L$.

Writing

$$(y-x)/y = \theta,$$

so that

$$x = y(1-\theta) = w_3(1-\theta),$$

we obtain

$$y = \frac{w_1}{w_2}(y-2x+1) = \frac{w_1}{w_2}(1-y+2y\theta)$$

$$y = \frac{w_3}{w_2}\left(1-\frac{x^2}{y}\right) = \frac{w_3}{w_2}(1-y\overline{1-\theta^2}),$$

so that

$$\left(\frac{w_2}{w_1}-\frac{w_2}{w_3}\right)y = \theta^2 y,$$

or

$$\theta^2 = \frac{w_2}{w_1}-\frac{w_2}{w_3}.$$

w_1, w_2, w_3 are proportional to the weights given by Ross, and hence θ can be determined. Ross's solution (*a*) then gives the value of

$$\Delta l - \theta\Delta L.$$

Thus, for 1912·1, the weights of Ross's solutions are 83 for Δl and 28 for ΔL in solution (*b*), and 661 for Δl in solution (*a*).

Therefore

$$w_1 : w_2 : w_3 = 83 : 28 : 661.$$

We derive

$$\theta = 0{\cdot}54$$

and conclude that, for 1912·1,

$$\Delta l - 0{\cdot}54\Delta\mathrm{L} = -0''{\cdot}65,$$

the constant being the longitude residual for this date in solution (*a*) in the sense Tabular *minus* Observed.

This is the quantity derivable from Ross's data with the greatest weight.

With the aid of the longitude errors of the Sun given in Table IV., the longitude errors of Mars can be derived.

If the observations were all secured near opposition, θ would have the value 0·66.

The values of θ resulting from the meridian observations near each opposition were computed from Ross's weights, and, using the Sun's errors from Table IV., the errors of Mars were derived. The results are given in Table VIII.

Ross's data only extend to the opposition of 1912, but accurate places of Mars up to the last opposition are provided by the heliometer observations secured at the Cape. Near opposition we have

$$\Delta\lambda = \frac{r}{\Delta}\Delta l - \frac{\mathrm{R}}{\Delta}\Delta\mathrm{L},$$

r, R being the radii vectores of Mars and of the Earth, and Δ the distance planet–Earth, so that $\Delta = r - \mathrm{R}$.

In the discussion of the Cape heliometer observations of Mars from 1899 to 1924,* Δl was derived from $\Delta\lambda$ by the formula

$$\Delta l = \frac{r - \mathrm{R}}{r}\Delta\lambda,$$

no account being taken of the longitude errors of the Sun. But it is evident that the quantities thus derived are not actually Δl but

$$\Delta l - \frac{\mathrm{R}}{r}\Delta\mathrm{L} = \Delta l - 0{\cdot}66\Delta\mathrm{L}.$$

The values of Δl, derived in the discussion of the Cape observations, were therefore corrected for ΔL from Table IV. and were further reduced to Ross's system so that the material should be homogeneous.

The results for Mars are given in Table VIII. The first column gives the date of opposition ; the second gives the value of θ derived from the weights given by Ross ; the third gives the derived longitude error in the sense Ross *minus* Observed. The fourth column gives the longitude residual derived from the heliometer observations. The fifth column

* Jones, *M.N., R.A.S.*, **85**, 853, 1925.

gives the weighted mean of the residuals in the two previous columns, the heliometer results receiving twice the weight of the meridian results on account of their freedom from systematic error. In the sixth column, the means by threes of the quantities in the fifth column are given, to smooth out the accidental irregularities. In the final column are given the values of $1{\cdot}88\times\Delta l$, 1·88 being the ratio of the mean motions of the Earth and Mars. The quantities in the last column should be comparable with the corresponding quantities derived for the Sun, Venus, and Mercury, if the residuals are due to variations in the rate of rotation of the Earth.

TABLE VIII.

Errors of Mean Longitude of Mars, 1825–1924.

(Ross – Observed.)

Date.	θ.	Δl (meridian).	Δl (heliometer).	Mean.	Δl (smoothed).	$1{\cdot}88\times\Delta l$.
		″	″	″	″	″
1835·0	+0·50	−0·31				
37·1	·49	+ ·12	..	..	−0·14	−0·26
39·2	·49	− ·22	..	..	− ·18	− ·34
1841·3	·63	− ·44	..	..	− ·34	− ·64
43·5	·48	− ·35	..	..	− ·32	− ·60
45·7	·56	− ·18	..	..	− ·11	− ·21
47·8	·51	+ ·21	..	..	·00	·00
49·9	·39	− ·04	..	..	+ ·10	+ ·19
1852·1	·55	+ ·14	..	..	+ ·11	+ ·21
54·2	·65	+ ·24	..	..	+ ·05	+ ·09
56·3	·60	− ·24	..	..	− ·21	− ·39
58·4	·55	− ·64	..	..	− ·42	− ·79
1860·5	·53	− ·37	..	..	− ·39	− ·73
62·7	·53	− ·17	..	..	− ·23	− ·43
64·9	·53	− ·16	..	..	− ·41	− ·77
67·0	·61	− ·89	..	..	− ·77	−1·45
69·1	·60	−1·26	..	..	− ·91	−1·71
1871·2	·64	− ·58	..	..	− ·85	−1·60
73·3	·59	− ·72	..	..	− ·51	− ·96
75·5	·51	− ·22	..	..	− ·24	− ·45
77·7	·53	+ ·22	..	..	− ·04	− ·08
79·9	·54	− ·13	..	..	− ·13	− ·24
1882·0	·55	− ·49	..	..	− ·34	− ·64
84·2	·57	− ·39	..	..	− ·27	− ·51
86·3	·62	+ ·07	..	..	− ·35	− ·66
88·3	·62	− ·72	..	..	− ·27	− ·51

TABLE VIII.—*continued.*

Date.	θ.	Δl (meridian).	Δl (heliometer).	Mean.	Δl (smoothed).	$1{\cdot}88 \times \Delta l$.
		″	″	″	″	″
1890·4	+ ·60	− ·14	..	..	− ·40	− ·75
92·7	·52	− ·33	..	..	− ·06	− ·11
94·9	·57	+ ·28	..	..	+ ·15	+ ·28
97·0	·57	+ ·50	..	..	+ ·27	+ ·51
99·1	·60	+ ·09	− ·02	+ ·02	+ ·23	+ ·43
1901·1	·53	+ ·26	+ ·14	+ ·18	+ ·02	+ ·04
03·2	·63	+ ·10	− ·25	− ·13	+ ·03	+ ·06
05·4	·59	+ ·26	− ·08	+ ·03	− ·17	− ·32
07·6	·54	− ·47	− ·40	− ·42	− ·26	− ·49
09·9	·54	− ·35	− ·43	− ·40	− ·50	− ·94
1912·1	·54	−1·09	− ·46	− ·67	− ·58	−1·09
14·2	..	..	− ·68	− ·68	− ·67	−1·26
16·3	..	..	− ·65	− ·65	− ·65	−1·22
18·3	..	..	− ·61	− ·61	− ·66	−1·24
1920·4	..	..	− ·71	− ·71	− ·71	−1·33
22·5	..	..	− ·80	− ·80	− ·71	−1·33
24·6	..	..	− ·62	− ·62		

The figures given in the last column are plotted in fig. 4. They form a much more irregular curve than in the case of Venus. This was to be anticipated, as Mars is not so suitable as Venus, on account of its smaller mean motion, for revealing the effects of variation in the rate of rotation of the Earth. Further, as Ross points out,* there are large systematic errors in the meridian observations of Mars. Our results for Venus, based almost entirely on the observations at Greenwich alone, are therefore better suited for revealing the residual fluctuations than the results for Mars, which are based upon the observations at several observatories between which there are large systematic differences. Nevertheless, in spite of its irregularity, the curve for Mars shows certain features common to the other curves : a pronounced drop about 1870, followed by a rise which from about 1895 is followed by a well-marked gradual fall. The latter feature, based mainly upon the heliometer observations, which are probably free from systematic error and are of small probable error, is the most clearly defined feature of the curve.

8. The evidence obtained in the present discussion on the question of the variability of the rate of rotation of the Earth is summarised in figs. 1 to 4, which represent the residual longitude errors of the Sun and of Mercury, Venus, and Mars, the residuals in the case of the planets being multiplied by the ratio of the mean motion of the Earth to the mean motion of the planet. If these residuals are due to slight changes

* *Astron. Papers of the American Ephemeris*, **9**, pt. ii., p. 268.

in the rotation period of the Earth, the four curves should theoretically be identical, assuming that the tabular longitudes and mean motions are free from error. But observations both of the Sun and of the planets are peculiarly liable to errors of a systematic nature, and the curves will be modified by these errors. Considering the smallness of the quantities under discussion, the general similarity between the four curves is very marked. They all show very clearly a gradual fall of about the same amplitude commencing near 1896, the rate of fall slowing down during recent years. All the curves agree in showing a minimum somewhere near 1870, preceded by a gradual fall from the beginning of the period discussed and followed by a gradual rise to about 1896, this rise being at first fairly rapid, then slowing down for a time and then again becoming more rapid a few years before the maximum. Not only are the curves qualitatively similar, but also the amplitude changes are approximately equal. The most irregular curve is the portion of the Mars curve which is dependent only upon meridian observations ; this is in accordance with expectation arising from the relatively slow mean motion of Mars.

The general similarity between the several curves may be expressed numerically in the form of a correlation coefficient. The correlation coefficients in the case of a planet and the Sun have been derived from the planetary longitude residuals as given in Tables V., VI., VII., and VIII., and longitude residuals for the Sun interpolated for the same epochs from Table IV. In the case of Mercury and Venus, the longitude residuals of Mercury from Table VI. were used in conjunction with longitude residuals for Venus interpolated from Table VII. The following values were derived :—

TABLE IX.

Correlation Coefficients.

Mercury (meridian observations)	and Sun	+0·54±0·11
Mercury (transits)	and Sun	+ ·85± ·05
Mercury (,,)	and Venus	+ ·82± ·06
Venus and Sun		+ ·83± ·03
Mars and Sun		+ ·86± ·03

The high values of these correlation coefficients give strong support to the view that the longitude fluctuations of the Sun, Mercury, Venus, and Mars are due to a common cause, and the approximate equality in amplitude of the reduced fluctuations is in favour of the supposition that the common cause is the variation in the rate of rotation of the Earth.

It appears that this cause alone is competent to account for the major portion of all the four curves throughout the period considered ; but even though the results for the different bodies had been discordant during the early portion of the period under discussion, the well-marked run in the residuals which commenced in every case about 1896 would in itself be sufficient to prove that a common cause was responsible, and

to indicate a change in the rotation period of the Earth as the operative cause.

Assuming, therefore, that the derived residuals are due to this cause modified by the accidental and systematic errors of observation, a composite curve can be constructed by combining the results from the four bodies with a rough system of weighting. For this purpose, smoothed curves were drawn to represent the run of the residuals as closely as possible, and from these curves the ordinates were read off at intervals of two years. The composite curve was obtained by giving a weight 3 to the Sun, 2 to Venus, and 1 each to Mercury (transits) and to Mars. In the case of Mercury, there are comparatively few points through which to draw a curve, rendering somewhat uncertain the run of the curve at intermediate points; otherwise a higher weight would have been given to the transit results. The longitude residuals of the Sun as given by the composite curve are tabulated in Table X.

TABLE X.

Errors of Sun's Longitude, 1836–1924 (*Tab.–Obs.*).

(Mean from Sun, Mercury, Venus, and Mars.)

	1830.	1840.	1850.	1860.	1870.	1880.	1890.	1900.	1910.	1920.
	″	″	″	″	″	″	″	″	″	″
+0	..	−0·13	−0·26	−0·54	−0·73	−0·23	−0·14	+0·04	−0·80	−1·17
2	..	− ·17	− ·30	− ·55	− ·57	− ·20	− ·03	− ·11	− ·90	−1·20
4	..	− ·20	− ·37	− ·60	− ·45	− ·18	+ ·09	− ·24	− ·97	−1·20
6	−0·03	− ·22	− ·43	− ·67	− ·35	− ·17	+ ·19	− ·43	−1·04	
8	− ·07	− ·24	− ·51	− ·77	− ·29	− ·16	+ ·14	− ·62	−1·11	

The residuals from the composite curve are plotted in fig. 7. They are in general satisfactorily small and oscillatory. Owing to the process of smoothing which has been used, the residuals are not of a purely accidental character, but it will be noticed that in no case is there a well-marked discordance persisting over a long period of years.

9. It is of interest to compare this composite curve with the curve given by the longitude residuals of the Moon. For this purpose the longitude residuals given by Dyson and Crommelin * were used. These residuals were obtained from the Greenwich observations of the Moon by comparing them with Brown's longitude, as corrected by Fotheringham. Fotheringham corrected the theoretical value of the secular acceleration used by Brown so as to fit the records of ancient eclipses, and this involved correlative changes in the mean motion and the great empirical term. In comparing observation with Brown's tabular longitude as thus revised by Fotheringham we are not making a comparison between observation and pure gravitational theory. It was first shown by Newcomb that there was an important long-period inequality in the Moon's longitude apparently not accounted for by gravitational theory, for which a purely empirical representation was

* *M.N., R.A.S.*, **83**, 359, 1923.

made. As modified by Fotheringham, this empirical term has a period of about 260 years and a semi-amplitude of 13″·6. Superposed on this are minor fluctuations of irregular period and of three or four seconds of arc amplitude. It is these minor fluctuations extracted from Dyson and Crommelin's paper which are plotted in fig. 6 for comparison with the composite curve representing the fluctuations in the Sun's longitude, which is shown in fig. 5. The scale of the curve representing the fluctuations of the Moon's longitude is adjusted to give an amplitude range approximately equal to that of the curve representing the Sun's longitude fluctuations.

A close similarity between the two curves is apparent. In fact, from 1870 onwards, the two curves are to all intents and purposes identical: previous to 1870, both curves show a gradual run down, but the curve for the Moon lies considerably below that for the Sun. This may possibly be due to systematic errors—personal and instrumental—affecting the earlier observations of the Sun and planets.

The well-marked general similarity between the run of the minor irregular fluctuations of the Moon's longitude and the composite curve derived from the Sun and planets is strongly in favour of the view that the minor fluctuations in the Moon's longitude are due to irregular variations in the Earth's rate of rotation.

It should be emphasised, however, that if comparison is made with the longitude of the Moon as given by gravitational theory with the great empirical term omitted, the total fluctuations in the Moon's longitude are not similar to those found for the Sun, Mercury, Venus, and Mars.

It remains to examine whether the ratio of the amplitude of the minor fluctuations in the Moon's longitude to the amplitude of the fluctuations in the Sun's longitude is in accordance with theory, when both are interpreted as arising from variations in the Earth's rate of rotation.

If ν, N, n denote respectively the mean motion of terrestrial rotation and of the orbital motion of the Earth and Moon, E, M denote the masses of the Earth and Moon respectively, a the radius of the Earth and R the radius of the Moon's orbit, then conservation of angular momentum gives, approximately,

$$\mathrm{E}k^2\nu+\mathrm{MR}^2n = \text{const.},$$

the effect of the obliquity of the ecliptic being neglected and the angular momentum of the Moon being relatively too small to take into account. $\mathrm{E}k^2 = \mathrm{C}$, the moment of inertia of the Earth about its axis of rotation.

When ν and n are varied, R will vary also so as to satisfy the relationship

$$n^2\mathrm{R}^3 = \mathrm{E}+\mathrm{M}.$$

It therefore follows that

$$\mathrm{E}k^2 \,.\, \delta\nu = \tfrac{1}{3}\mathrm{MR}^2 \,.\, \delta n$$

or

$$\frac{\delta n}{n} = 3\frac{\mathrm{E}}{\mathrm{M}} \cdot \frac{k^2}{\mathrm{R}^2}\frac{\nu}{n} \cdot \frac{\delta\nu}{\nu}$$

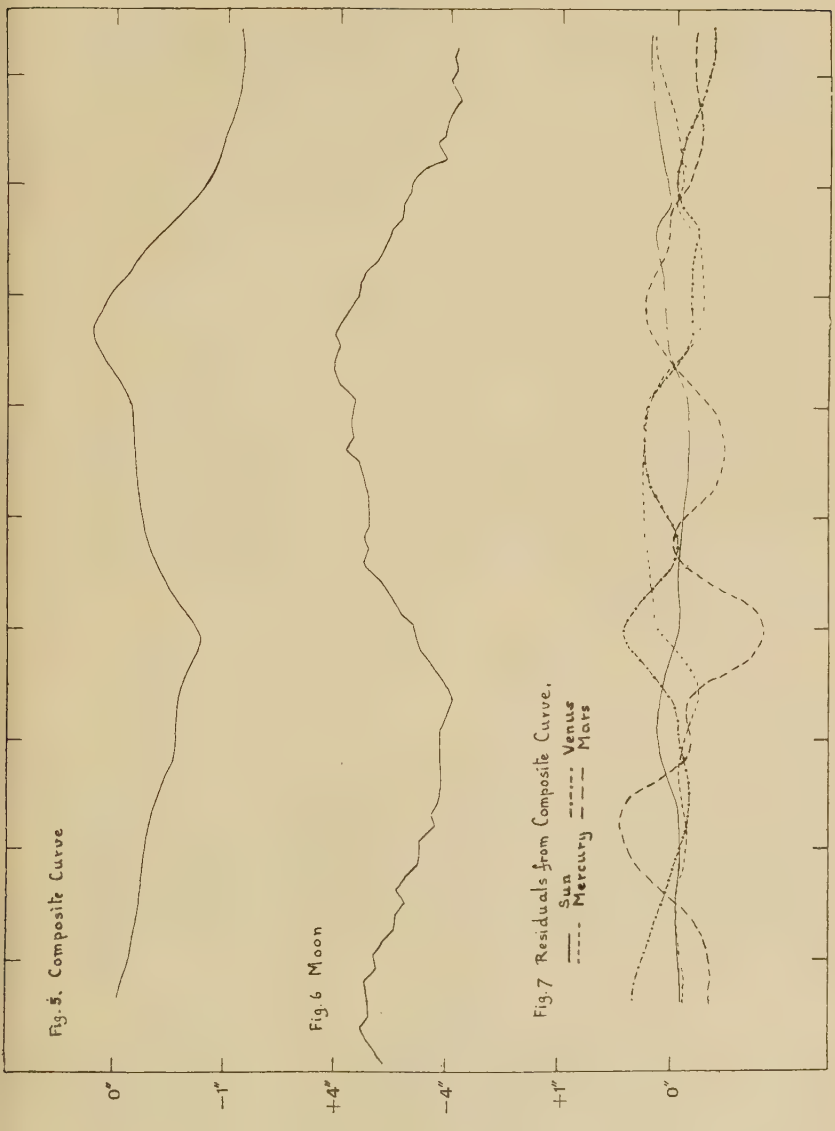


Fig. 5. Composite Curve
Fig. 6 Moon
Fig. 7 Residuals from Composite Curve.
Sun
Venus
Mercury
Mars
0"
−1"
+4"
−4"
+1"
0"
1840
1850
1860
1870
1880
1890
1900
1910
1920

Now $k^2 = 0{\cdot}334a^2$,* $a/\mathrm{R} = 3422''{\cdot}7$, $\mathrm{E}/\mathrm{M} = 81{\cdot}53$, $\nu/n = 27{\cdot}32$, and therefore

$$\frac{\delta n}{n} = 0{\cdot}61 \times \frac{\delta \nu}{\nu}.$$

Further, the changes in n and ν are connected with the errors in longitude, δl and $\delta\mathrm{L}$, of the Moon and Earth respectively by the relations

$$\frac{d}{dt}(\delta l) = n\left(\frac{\delta \nu}{\nu} - \frac{\delta n}{n}\right) = 0{\cdot}39 \times n \,.\, \frac{\delta \nu}{\nu}$$

$$\frac{d}{dt}(\delta \mathrm{L}) = \mathrm{N} \,.\, \frac{\delta \nu}{\nu}.$$

Thus

$$\frac{d}{dt}(\delta l) = 0{\cdot}39 \times \frac{n}{\mathrm{N}} \,.\, \frac{d}{dt}(\delta \mathrm{L})$$

or

$$\frac{d}{dt}(\delta l) = 5{\cdot}2 \times \frac{d}{dt}(\delta \mathrm{L}),$$

since $n/\mathrm{N} = 13{\cdot}4$.

If we consider the changes in the longitude errors between 1897 and 1924, during which period the most rapid change occurs, we may take $d(\delta l)/dt$ and $d(\delta \mathrm{L})/dt$ as proportional to the amplitudes of the changes in the longitude residuals during this period. Comparing the longitude change for the Moon with the changes found for Mars, Venus, Mercury (reduced to the basis of the Earth's mean motion) and the Sun, we find the following ratios :—

For the Sun	a ratio of	6·7
Mercury	,,	6·7
Venus	,,	5·1
Mars	,,	5·0
composite curve	,,	6·1.

These ratios are in tolerably good agreement with the theoretical figure of 5·2. Bearing in mind the liability of observations of the Sun and planets to systematic errors and the smallness of the longitude residuals for these bodies, the agreement is as good as would be anticipated. We conclude therefore that the minor fluctuations in the longitude of the Moon can be attributed to irregular variations in the rate of rotation of the Earth.

Summary.

1. The longitude errors of the Sun, Mercury, Venus, and Mars during the period 1836 to 1924 have been investigated. Except in the case of Mars and Mercury, the discussion was based mainly upon the Greenwich series of observations. Most of the data for Mars were obtained from Ross's discussion of the elements. For Mercury, Innes's

* Jeffries, *The Earth*, p. 190.

data from the transits were utilised in conjunction with the data given by the meridian observations.

2. It is found that the residual longitude fluctuations of the four bodies are generally similar. They all show a gradual drop to about 1870, followed by a rise, more rapid at first and then slowing down continuing until about 1897. This in turn is followed by a progressive fall, tending to slow out in recent years. The amplitudes of the longitude changes are approximately proportional to the mean motions of the bodies.

3. The longitude fluctuations in the four bodies can be attributed to irregular variations in the rate of rotation of the Earth.

4. A composite curve obtained by combining the results from the four bodies shows very marked similarity to the curve of the minor fluctuations in the Moon's longitude. The ratio of the amplitudes of the two curves agrees fairly well with the theoretical value derived on the hypothesis that the minor longitude fluctuations of the Moon are also due to irregular variations in the rate of rotation of the Earth.

5. It is therefore concluded that during the period 1836 to 1924 the longitude fluctuations of the Sun, Mercury, Venus, and Mars, and also the minor longitude fluctuations of the Moon, can be attributed, within the limits of error of the observations, to changes in the rate of rotation of the Earth.

Royal Observatory, Cape of Good Hope :
1926 *June* 15.

Note on Three Peculiar Spectra. By J. S. Plaskett, F.R.S.
(Plates 1–3.)

In the course of the regular observing programme of B- and O-type stars at Victoria, two of the stars with peculiar spectra, H.D. 45910 and H.D. 50820, described in this paper were discovered. The other, υ Sagittarii, was undertaken because of the enormous masses of 260 and 54 times the sun ascribed to the system by Ludendorff on somewhat uncertain evidence, but observations were continued on account of the peculiarities of the spectrum and their important astrophysical bearing. The three spectra, however, are so similar in many respects that their description can best be combined into one paper. A full description and discussion with complete illustrations will appear shortly as vol. iv., No. 1, *Publications of the Dominion Astrophysical Observatory*, but in the meantime a short note on the principal features of these remarkable stellar spectra will be of interest.

The three spectra are similar in complexity, each having complete sets of lines characteristic of at least two distinct spectral types. What appears to be the fundamental spectrum in each star, as it is always present with apparently unchanged intensity, is of type B2 or B3, with all the lines of helium, ionised carbon, magnesium, etc., present in stars of this type. But superposed on this helium spectrum is, in each case,

a many-lined metallic spectrum of type A2 which, in one of the stars appears and disappears, in another is certainly and in the third probably variable in intensity. Moreover, in each spectrum the hydrogen lines have a complex, peculiar, partly emissive and variable structure somewhat analogous to the novæ. These similarities are, however, contrasted by some curious differences which can best be referred to in the separate descriptions.

v Sagittarii.

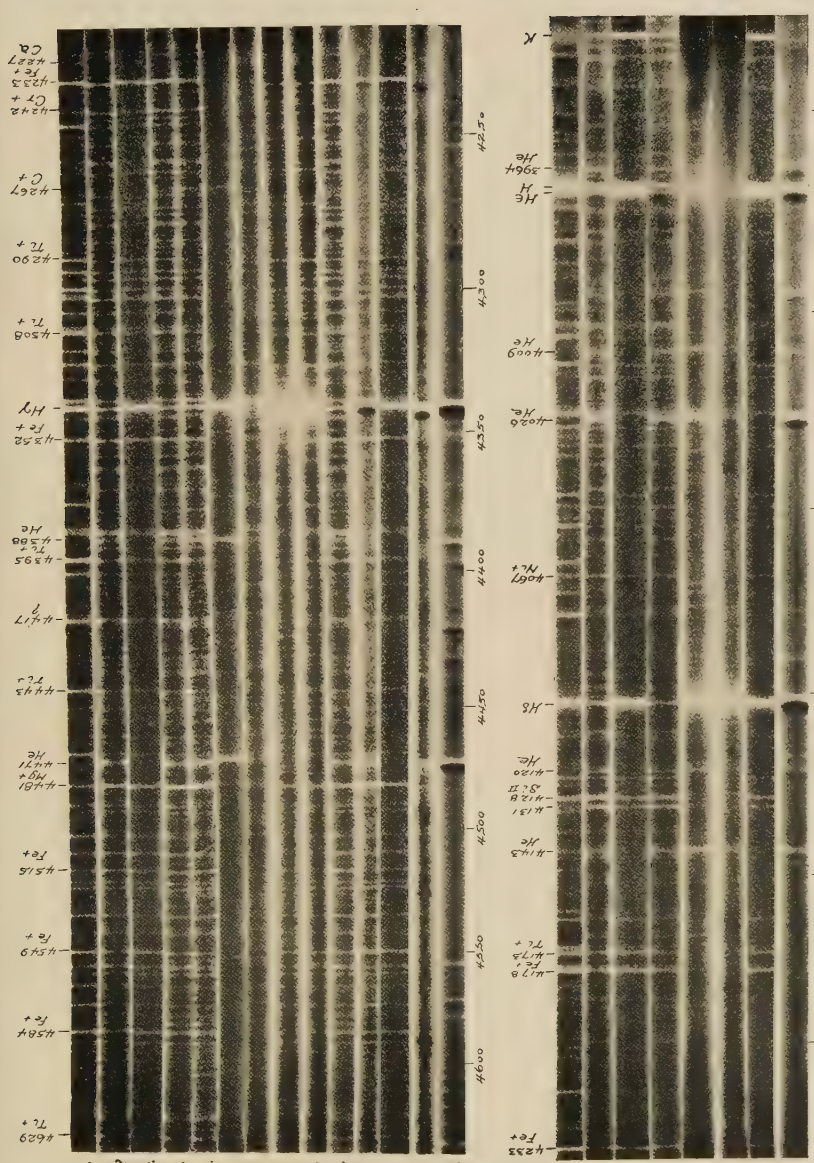
This complex and very peculiar spectrum was classified B8p, F2p at Harvard, was assumed to be composite and due to two or three bodies, possibly something of the order of β Lyræ. It will be best to describe it under three headings.

(*a*) *The Metallic Spectrum.*—The metallic spectrum is unlike any other observed here, is practically entirely a spectrum of enhanced lines of the elements Fe, Ti, Cr, Ni, Sr, Mg, Ca, and possibly V, Mn, Al, while the Si II lines 4128, 4131 are the strongest lines in the spectrum. All the lines of α Cygni are present as well as a great many others and with greatly increased strength, having the "c" characteristics to a supernormal extent, indicating high luminosity and low density. Although the great number of lines makes the spectrum appear of F-type, there are no arc lines, and the relative strength of H and K place it definitely in class A2. The metallic spectrum appears variable in intensity, the lines occasionally becoming relatively weaker.

(*b*) *The Helium Spectrum.*—The helium spectrum, apparently of constant intensity in all phases, is superposed on the metallic spectrum, has much stronger and sharper helium and C4267 lines, evidently sharing the "c" characteristic with the metallic lines, than in the normal B2 or B3 spectrum to which type it most closely corresponds. The strength and sharpness of metallic and helium lines make it improbable that they arise from separate stars, while the identical velocity displacements of both sets of lines in all orbital phases conclusively prove that both helium and metallic lines originate in one stellar atmosphere. The simultaneous presence of lines corresponding normally to stellar temperatures so different as 15,000° and 9000° may be due to a supernormal abundance of helium or more probably to the star being an exaggerated form of pseudo-cepheid or giant.

(*c*) *The Hydrogen Spectrum.*—The hydrogen spectrum is also very peculiar, variable, partly emissive, and rather suggestive of novæ. It consists, at each of the Balmer lines, β, γ, δ, and ϵ, measured, of a strong absorption line in the normal position with absorption components, always to the violet of the primary and never to red, variable in number and intensity and at velocity displacements of approximately -150, -200, -300 km. per second from the primary. Both the primaries and all the satellites, which are shown to be due to hydrogen and not unidentified metallic lines, partake of the same orbital velocity changes as the metallic and helium lines, thus proving the single stellar origin of the whole complex spectrum.

The component at -300 km., which is the most persistent of the


ε AUR.
υ SGR
α CYG.
υ SGR.
υ SGR.
199140
50820
o VIR.
16 ORI.
υ SGR.
45910
α CYG.
45910
P CYG.
4629 Ti +
4584 Fe +
4549 Fe +
4515 Fe +
4481 Mg + 4471 He
4443 Ti
4417 ?
4395 Ti + 4388 He
4352 Fe +
Hγ
4308 Ti +
4290 Ti +
4267 Ca +
4242 Cr +
4233 Fe + 4287 Ca
4250
4300
4350
4400
4450
4500
4550
4600
4233 Fe +
4178 Fe, Ti +
4143 He
4131 Si II 4128
4120 He
Hδ
4067 Ni +
4035 He
4009 He
He + H 3984 He
K
4200
4150
4100
4050
4000
3950

PHASE
31
47
55
122
125
130
133
137
V.E.L.
-141, -303
-215
-195
-283
-293
-321
-304
-156, -294
Hβ
-1700
-2000
-3000
PHASE
1
11.7
5.5
6.7
120
12.5
13.0
13.3
VEL
-169, -315
-137, -233
-198, -325
-182, -328
-190
-165 -275
-189, -308
-168 -294
Hγ
-1700
-1800
-3000

1922 OCT. 9
1923 MAR. 2
" MAR. 17
" MAR. 25
" MAR. 27
" MAR. 28
" MAR. 29
" MAR. 31
" APR. 1
" APR. 10
" APR. 25
" NOV. 5
1924 MAR. 16
" NOV. 27
1925 MAR. 26
1926 MAR. 11
H δ
H γ
H ε
H
K
4000
4100
4200
4300
3950
4000
4050

satellites, is obviously the supposed "doubled" line on which Ludendorff's enormous masses were based. It has, however, nothing to do with orbital motion, as it is always at −300 km. from the primary at all phases in the orbit, whereas it should vary between −300 and +300 if it were due to orbital doubling. Although Ludendorff's enormous masses are thus quite illusory, there seems no doubt from Wilson's mass-function of 1·58 ⊙ and the super "c" character of the spectrum that the mass is over 20 times the sun.

H.D. 50820.

This spectrum is remarkably similar to υ Sagittarii in its complex or composite character, but the star is not a binary and hence the critical criterion of orbital oscillations cannot be applied as in υ Sagittarii. The metallic lines are more nearly of the F-type, the arc and enhanced lines nearly equal in strength and all somewhat diffuse. Further, the metallic lines become very weak and practically disappear to the violet of λ4150 where the spectrum is similar to the average B3. The helium lines of class B3, always present and apparently constant in intensity, are more diffuse than in υ Sagittarii, but sharper than the average B3. The helium lines agree in velocity with the metallic lines, and this, with the improbability of there being an F and a B3 star of the same magnitude and velocity in the same line of sight and the similarity to υ Sagittarii, point to a common origin. On the other hand, the diffuseness and weakness of the lines, the fading to violet of the metallic spectrum, and the apparently much denser atmosphere are opposed to an origin in one stellar atmosphere. The hydrogen spectrum is more peculiar even than in υ Sagittarii, having absorption satellites at H_β and H_δ, but a wide structureless absorption band at H_γ. The threefold similarity of H.D. 50820 and υ Sagittarii, however, seems to point to some similarity in physical constitution.

H.D. 45910.

There are similarities also in H.D. 45910 to the other two stars in having a constant helium spectrum, an A2 metallic spectrum, and peculiar hydrogen lines. This star was the first one of the three observed, its peculiar character being independently discovered here on 1922 October 9, although previously observed at Mount Wilson. A preliminary description of its remarkable variable spectrum was given in *Pub. A.S.P.*, **35**, 145, 1923, but it has since been followed for three seasons and has shown great changes in its spectrum. The fundamental spectrum always present and apparently constant in character and intensity is of type B3 with diffuse helium lines, while the hydrogen lines are of the P Cygni type variable in position and character. When first observed the continuous spectrum appeared irregular, and later developments showed this irregularity to consist of incipient emission at the enhanced iron lines 5018, 4924 (which are occasionally of the P Cygni character), 4584, 4549, 4352, 4303, 4233. This emission waxed and waned, frequently showing beginnings of absorption at 4352, 4233,

while this absorption has on four separate occasions developed into a complete enhanced "c" metallic-absorption spectrum, an almost exact replica of α Cygni. The absorption stage is evidently very transitory and has only been observed at full strength in 1923 March. In addition to the remarkable change in type there is an observed variation in velocity of the metallic absorption lines of at least 70 km., but its transitory character did not allow detection of periodicity while the measures of helium absorption are too uncertain. The P Cygni absorption and emission lines of hydrogen, however, show a cyclic change of about 120 km. in a period of about 235 days, but this could not be correlated to the changes in the infrequently appearing metallic lines. This remarkable variable spectrum has analogies to the novæ in the P Cygni hydrogen lines, the transient appearance of α Cygni absorption, and the rapidly changing character of its spectrum.

Note on Dr. Pokrowsky's Proposal for Determining the Angular Diameter of a Star. By A. S. Eddington, F.R.S.

In *Astrophysical Journal*, **36**, 156, a method of determining the angular diameter of a star by means of elliptically polarised light was proposed by S. Pokrowsky. I know that many readers have been attracted by this ingenious proposal. I myself reviewed it favourably at the time in an unsigned note in the *Observatory*. Some years later, however, I came to the conclusion that there was a flaw in the method.

As the difficulties of making the proposed instrument on a large enough scale were considerable, it seemed at the time unlikely that the method would be given a serious practical trial. But the success of the Michelson interferometer has revived interest in the proposal, and I have learned recently that the construction of such an instrument has actually been planned. The flaw in the theory is rather subtle and seems generally to have escaped detection; so I have been urged to publish this note.

The essential principles of the method may be summarised as follows. As in Michelson's interferometer two widely separated apertures are used and the two beams through these apertures are brought into superposition. But before superposition the beams are polarised in perpendicular planes;* so that instead of the ordinary interference we obtain *plane* polarised light if the phase-difference is 0 or $\frac{1}{2}\lambda$, *circularly* polarised light if it is $\frac{1}{4}\lambda$ or $\frac{3}{4}\lambda$, and *elliptically* polarised light for intermediate values. Plane polarised light can be completely extinguished by a Nicol prism, but any elliptical polarisation will betray itself by incomplete extinction of the image. The idea is that with a point source of light the phase-difference can be made zero and the image completely extinguished; but with a finite disc the phase-difference will not be zero for all points of the disc simultaneously,

* They are polarised initially in the same plane and one beam is then rotated by a half-wave plate.

and therefore the image will not wholly disappear. By measuring the residual intensity of the image the angular diameter is to be found. The presumed advantage over Michelson's method is that the result depends on measurement of the intensity of the whole image (a measurement capable of great sensitivity) instead of an examination of the minute interference pattern.

It can be seen that there must be something wrong with the proposal in the following way. The centre of the disc is supposed to contribute zero intensity to the image, whilst points on the periphery distant only 0″·003 (for a first magnitude star, *loc. cit.*, p. 166) give observable brightness. In other words, a displacement of 0″·003 of the light source relative to the instrument causes the extinguished image to become visible; in fact a displacement of about 0″·5 would restore full intensity. Surely it is incredible that in an instrument of this kind a star image should pop in and out as the result of displacements of the order of a second of arc. (Even if this did occur it would render the method impracticable; that is to say, if we observed an image of a certain intensity we could not tell whether that indicated a stellar radius of 0″·003 or a point source viewed 0″·003 out of alignment. The question of securing alignment does not seem to be discussed in the paper.)

The fallacy arises in this way. The intensity which is calculated by Pokrowsky is not that of the whole star image, but intensity at a particular point in the focal plane. If the star is slowly displaced relatively to the axis of the instrument its image does not pop in and out, *but bright and dark interference bands pass across the point under consideration.*

The star in fact gives a system of interference fringes very similar to that shown in Michelson's interferometer, but modified by the polarising apparatus. No particular alignment of the instrument is required, the centre of the fringe system being always formed at the point where there is no phase-difference, *i.e.* coincident with the geometrical image of the source. As the source is displaced the fringe system is displaced without modification following the motion of the geometrical image. Just as in Michelson's method finite diameter of the source will modify the interference pattern (owing to superposition of fringe systems corresponding to each point of the disc), and the dark fringes will become less black. The calculation in Pokrowsky's paper gives the residual intensity in the central dark fringe. But, of course, it is not the intention of the method to study the interference pattern, since for this purpose the polarisation is a pointless complication and the larger Michelson instrument is obviously superior. The proposed measurement is that of the whole intensity of the image, *i.e.* the intensity integrated over the whole diffraction pattern. This we now see is simply additive for the individual light sources, whether they are concentrated at a point or spread over a disc, so that the measurement gives no clue to the diameter of the star.

The actual proposal is to examine the emergent light with a doubly refracting prism whose principal planes are at 45° to the original planes of polarisation. It is contemplated that for a point source one emergent

image will completely disappear and the other be of full intensity. According to the discussion above the image will not disappear. There will be a dark spot at the centre, and dark rings corresponding to retardations $\frac{1}{2}\lambda$, λ, $\frac{3}{2}\lambda$, etc.; but intermediately there will be bright rings. The ratio of intensities of the two images could be calculated without much difficulty; it will be the same for a point source and for a finite source.

The inadequacy of the treatment in the original paper arises because it assumes that (for a point source) plane waves emerge from the two apertures, travelling in a particular direction. But owing to diffraction waves are travelling in various directions inclined at angles small from the ordinary standpoint, but large in comparison with the angles (such as 0″·003) occurring in the investigation.

It is an ungrateful and ungracious task to show the breakdown of an elegant and attractive proposal which by its remarkable ingenuity seemed to deserve success. But there seems to be no doubt of the fallacy, and it is desirable that this should be known in order to avoid waste of time and money.

Note on the Internal Densities and Temperatures of the Stars.

By J. H. Jeans, Sec. R.S.

1. Current estimates of the density and temperature in stellar interiors are mainly based on the formulæ and calculations of Eddington. These were obtained through speculative assumptions (generally $k\eta$ = cons.) which led to λ, the ratio of gas-pressure to pressure of radiation, having a constant value throughout the star. So long as the true value of k was unknown such calculations might be regarded as legitimate speculation, but now that Kramers' formula for k is available, they ought to be superseded by calculations based on this formula. Such calculations are given in the present note.

2. Starting from Kramers' formula I have calculated * the actual value of λ, the ratio of gas-pressure to pressure of radiation. In the simplest case, in which energy is generated uniformly throughout the star's mass, I have shown (*loc. cit.*, p. 203) that λ varies as $T^{\frac{1}{2}}$ in the interior of a star of very small mass, and as $T^{\frac{1}{2}}$ in a star of very great mass. This shows that λ may vary by a factor of from 10 to 100 or more in the interior of a single star and emphasises the need for revising estimates which were originally based on the supposition that λ was constant throughout the interior of each star.

3. Let p_G, p_R denote gas-pressure and pressure of radiation so that

$$p_G = \frac{R}{m\mu}\rho T, \qquad p_R = \tfrac{1}{3}aT^4 \quad \text{and} \quad p_G = \lambda p_R . \qquad . \qquad (1)$$

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* *M.N.*, 1925 January, p. 199.

Introduce a new subsidiary temperature T′ such that

$$p_G+p_R = \frac{R}{m\mu}\rho T' \quad \text{and} \quad T' = T\left(1+\frac{1}{\lambda}\right) \quad . \quad . \quad (2)$$

Then the dynamical equations of equilibrium are

$$\frac{\partial}{\partial r}\left(\frac{R}{m\mu}\rho T'\right) = \rho\frac{\partial V}{\partial r} \quad . \quad . \quad . \quad . \quad (3)$$

The relation $p_G = \lambda p_R$ may be written in the form

$$\frac{R}{m\mu}\rho = \tfrac{1}{3}a\lambda T^3 = \tfrac{1}{3}a\frac{\lambda^4}{(1+\lambda)^3}T'^3 \quad . \quad . \quad . \quad (4)$$

In a star of very small mass, λ is very large, and I have shown that λ varies as $T^{\frac{1}{4}}$, or, what is the same thing in this special case, as $T'^{\frac{1}{4}}$. Hence equation (4) shows that, throughout, the star

$$\rho \propto T'^{3\cdot25} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The build of a star of very small mass is accordingly determined by equations (3) and (5). Its distribution of density is seen to be the same as that of a star in adiabatic equilibrium with γ, the ratio of the specific heats, equal to 1·308. The solution is given by Emden's standard solution with n=3·25.

In a star of very large mass, λ is very small, and I have shown that λ varies as $T^{\frac{1}{2}}$. Equation (2) now shows that T′ varies as $T^{\frac{1}{2}}$, so that λ varies as T′. Since λ is very small, equation (4) shows that in this case

$$\rho \propto T'^7 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and again the distribution of density is that of a star in adiabatic equilibrium, the solution being Emden's solution with $n = 7$.

Thus stars of very small mass and of very large mass are arranged according to the law

$$\rho \propto T'^n \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where n has the values 3·25 and 7 respectively. Let us assume that intermediate values of n in equation (7) give a fair approximation to the arrangement in stars of intermediate mass.

4. Emden * has tabulated numerical solutions for the cases of n = 3, 4, 4·5, 4·9, and 5, but intermediate solutions can only be found by interpolation. In Table I. the entries in the second and third columns in Roman type are values of r_1 and $\rho_0/\bar{\rho}$ tabulated by Emden, while the entries in italic type are values I have calculated by interpolation. The interpolation is simple if it is noticed that $1/r_1$ runs very smoothly with n, while $r_1^3(\bar{\rho}/\rho_0)$ runs very smoothly indeed, being nearly independent of n.

* *Gaskugeln*, p. 78 et ff.

TABLE I.

n.	r_1.	$\rho_0/\bar{\rho}$.	$\frac{1}{(n+1)r_1^2}\left(\frac{\rho_0}{\bar{\rho}}\right)$.	Application.
2·5	5·42	24·08	0·234	(Emden's diatomic model.)
3·0	6·90	54·36	0·285	(Eddington's model ($k\eta$ = cons.).)
3·25	*8·00*	*88*	0·323	Actual star, very small mass.
3·5	*9·52*	*152*	0·374	Actual stars.
4·0	15·00	623	0·554	
4·5	32·14	6377	1·123	
4·9	169·47	934,800	5·518	
5·0	∞	∞	∞	
7·0	∞	∞	∞	Star of infinite mass.

The meaning of the fourth column will become apparent later.

5. From Emden's formulæ, the value of T′ at the centre of a star which is arranged as determined by equations (3) and (7) is given by

$$T_c' = \frac{3\gamma\mu}{(R/m)(n+1)r_1^2}\left(\frac{\rho_0}{\bar{\rho}}\right)\left(\frac{M}{r}\right) \quad . \quad . \quad . \quad (8)$$

so that, from equation (2), the central temperature T_c is given by

$$T_c = \frac{3\gamma\mu}{(R/m)(n+1)r_1^2}\,\frac{\lambda_c}{1+\lambda_c}\left(\frac{\rho_0}{\bar{\rho}}\right)\left(\frac{M}{r}\right) \quad . \quad . \quad (9)$$

Inserting the well-known numerical values $R/m = 8{\cdot}26\times10^7$, $\gamma = 6{\cdot}66\times10^{-8}$, this becomes

$$T_c = 69{\cdot}6\times10^6\frac{\lambda_c}{1+\lambda_c}\left[\frac{1}{(n+1)r_1^2}\left(\frac{\rho_0}{\bar{\rho}}\right)\right]\left(\frac{M}{r}\right)\mu \quad . \quad . \quad (10)$$

where M and r are measured in terms of the sun's mass.

In this formula the last term μ is the effective molecular weight, which can be guessed at with fair accuracy. The next term M/r can be determined observationally. The value of λ_c, the ratio of gas-pressure to radiation-pressure at the star's centre, depends mainly on the star's mass, and is easily estimated. Finally, the quantity in square brackets is that tabulated as a function of n in Table I. It is easily evaluated when n is known; the main difficulty is to determine n for a given star.

6. When η, the average rate of generation of energy inside a sphere of radius r, varies as T^l, the value of λ is given to a good enough approximation * by

$$\lambda(\lambda+1) = K p_R^{\frac{1}{2}(\frac{1}{2}-l)},$$

where K is a constant.

Putting $p_R = \frac{1}{3}aT^4$, and using equation (2), this becomes

$$\lambda^{\frac{1}{2}+l}(\lambda+1)^{\frac{3}{2}-l} = K(\tfrac{1}{3}a)^{\frac{1}{2}(\frac{1}{2}-l)}T'^{(\frac{1}{2}-l)} \quad . \quad . \quad (11)$$

* *M.N.*, **85** (1925), 196.

With this value for λ, the dependence of ρ on T′ is given by equation (4), namely

$$\rho \propto \frac{\lambda^4}{(1+\lambda)^3} T'^3 \quad . \quad . \quad . \quad . \quad (12)$$

This variation is not of course of the simple form

$$\rho \propto T'^n \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which leads directly to Emden's solution. But although we cannot make formulæ (12) and (13) agree for all values of T′, we can make the two curves coincide for any two selected values of T′. As the temperature range inside the greater part of a star's mass is in no case enormously great, we should do well to take the two selected values of T′ fairly close together. A convenient plan, which seems to give very fair accuracy, is to suppose these two points to be coincident at a value of about 0·8 T_c', where T_c' is the value of T′ at the centre of the star. In actual cases that I have tried this has been found to give fair accuracy within the range 0·4 T_c' to T_c', within which the main mass of the star lies.

Analytically, the equations to be satisfied are (*cf.* equations (12) and (13)),

$$\frac{\lambda^4}{(1+\lambda)^3} = BT'^{(n-3)} \quad . \quad . \quad . \quad . \quad (14)$$

where B is a constant (the ratio of the constants on the right of equations (12) and (13)), and the equation obtained by differentiating this with respect to λ, namely

$$\frac{4}{\lambda} - \frac{3}{1+\lambda} = \frac{n-3}{T'} \frac{\partial T'}{\partial \lambda}.$$

Using the value of T′ in terms of λ given by equation (11), this last equation becomes

$$\frac{4}{\lambda} - \frac{3}{1+\lambda} = \frac{n-3}{1-2l}\left[\frac{1+2l}{\lambda} + \frac{3-2l}{\lambda+1}\right].$$

This may be regarded as giving the value of n. Simplifying, we have

$$n = 3 + (1-2l)\left(\frac{4+\lambda}{1+4\lambda+2l}\right) \quad . \quad . \quad . \quad (15)$$

In this λ must be given the value appropriate to T′ = 0·8 T_c' or thereabouts, and this is only slightly less than the value of λ at the centre of the star.

7. In the special case in which energy is generated uniformly throughout the star, $l = 0$ and the equation reduces to

$$n = 3 + \frac{4+\lambda}{1+4\lambda} \quad . \quad . \quad . \quad . \quad (16)$$

which includes the results obtained in § 3 as special cases.

With this value of n it is easy to obtain the values of T_c and ρ_c

for any star we please. Examples are given in the following table, the values of λ and T_c being evaluated for the standard effective molecular weight $\mu = 2{\cdot}5$. For other values of μ, T_c is proportional to μ, except that the value of λ will also be somewhat different.

TABLE II.

Central Densities and Temperatures.

Star.	Mass. ($\odot = 1$).	λ ($\mu = 2{\cdot}5$).	n.	$\bar{\rho}$.	ρ_c.	r ($\odot = 1$).	T_c ($\mu = 2{\cdot}5$).
Plaskett's star	? 86	0·21	5·3	0·014	v. great	9·0	v. great
V Puppis	19·2	0·5	4·5	0·07	400	7·6	165×10^6
u Herculis	7·6	1·1	3·94	0·14	70	4·3	86×10^6
Sirius	2·45	3·1	3·53	0·85	140	1·58	79×10^6
Sun	1·00	10·6	3·34	1·42	150	1·00	54×10^6
60 Kruger A	0·25	130	3·26	12·7	1100	0·30	47×10^6
60 Kruger B	0·20	200	3·25	83	7300	0·15	75×10^6
α Orionis	? 40	0·33	4·9	0·000002	2	300	32×10^6
Capella A	4·18	1·9	3·69	0·005	1	11	19×10^6
α Cent. B	0·97	11·0	3·33	0·13	14	2·1	25×10^6

The entries *v. great* for the central temperature and density of Plaskett's star indicate that our formula gives infinite values owing to n being greater than 5. This may mean merely that our analysis is not delicate enough to determine these quantities. It might also mean that the central temperature is so high that the atomic nuclei are partially dissociated. This would bring about a substantial lowering in the value of μ which would speedily increase λ and reduce the value of n to below 5. Whether this is the correct interpretation or not, it seems impossible to determine ρ_c and T_c for this star with any reasonable degree of accuracy.

We see at once that the central densities of stars are very much higher than has generally been supposed. The central temperatures are also higher and vary widely among themselves. The upper group of stars in the table are all on or close to the main sequence, and the central temperatures of these stars vary sufficiently to dispose of the suggestion advanced by H. N. Russell,* that all stars on the main sequence have very approximately the same central temperature (which, using Eddington's model, he estimated at about 32×10^6). Different data might give somewhat different values for T_c and ρ_c from those given in the table, but it hardly seems possible that any permissible choice of data should do away with the difference in central temperature between, say, V Puppis and the sun.

8. When once it is established that T_c varies considerably along the main sequence, it becomes necessary to abandon Russell's further suggestion that a star's generation of energy is a consequence of its central temperature attaining a certain definite critical temperature.

* *Nature*, 1925 August 8.

I have recently advanced the alternative suggestion * that the main sequence is determined by the condition that a substantial fraction of the atoms near a star's centre are stripped bare, or nearly bare, of electrons. The original reasons for this suggestion were purely dynamical, arising out of the necessity of keeping the stars dynamically stable, and need not be discussed here. We may, however, examine to what extent the suggestion accords with the results obtained in the present paper.

The energy W required to strip the last 1-quantum-orbit electron from a nucleus of atomic number N is readily shown to be

$$W = \frac{2\pi^2 me^2 E^2}{h^2} = 21{\cdot}8N^2 \times 10^{-12} \text{ ergs.}$$

At any temperature T, the quantum of radiation of wave-length λ_{max} is $6{\cdot}8 \times 10^{-16}T$ ergs. With T equal to 100 million degrees, this quantum would strip bare all nuclei of atomic number less than 56. But at 100 million degrees there are an appreciable number of quanta having four times this energy, and so capable of stripping bare all atoms of atomic numbers up to 112. On the whole the temperatures just calculated support the interpretation of the main sequence I had previously put forward, and accord with the result I have recently obtained from a study of stellar opacity,† namely, that the atoms of the stars have higher atomic weights than has generally been supposed, and that the mean atomic number steadily decreases as a star gets older.

My suggestion required that atoms ceased to generate energy when they were stripped bare of electrons. Thus the generation of energy in the central regions of main-sequence stars would be very slight. The central regions of these stars would approximate to stagnant masses of inert, completely dissociated matter, while the main generation of energy would occur further out. Evidence in favour of this is found in the circumstance that stars on the main sequence are unduly luminous for their masses and observed spectral types.‡ If this is accepted, all of the foregoing calculations which refer to main-sequence stars need modification on account of the radiation not being generated uniformly throughout the star's mass.

9. To examine the nature of this modification we return to the general equation (15) which gives the value of n corresponding to a non-uniform generation of energy such that η varies as T^l. To represent a generation of energy which takes place mainly away from the star's centre, we must assign a negative value to l. The law $\eta \propto T^l$ need not hold in the star's outermost layers of all. The main mass of a star, and probably also its main generation of energy, is confined to temperatures between T_c and $\frac{1}{3}T_c$. If the T^l law is assumed to hold within this range, the value $l = -1$ would imply that the generation of energy per unit mass at the star's centre is only one-third of its average value throughout the star.

* *M.N.*, **85** (1925), 914.
† *Ibid.*, **86** (1926), 561.
‡ *Ibid.*, **85** (1925), 199.

If we adopt the value $l = -1$, equation (15) becomes

$$n = 3 + \frac{12+3\lambda}{4\lambda-1} \quad . \quad . \quad . \quad . \quad (17)$$

in place of our previous equation (16). Even for a star of very small mass ($\lambda = \infty$), the value of n is now 3·75, and Table I. now shows that the previous values for ρ_c (which were calculated for $n = 3{\cdot}25$) must be increased nearly four-fold, while the calculated central temperatures must be increased by about 40 per cent. For more massive stars the increases are incomparably greater; the value of n reaches the value 5, which corresponds to infinite central density and temperature, as soon as λ falls to 2·8, and this corresponds to a mass of only $2{\cdot}7\odot$. Thus giant stars on the main sequence may have enormous central temperatures and densities, far higher than those shown in Table II., while the central temperature and density even of Sirius ($M = 2{\cdot}45\odot$) may be very high indeed. With the value $l = -1$, equation (17) shows that n for the sun would be 4·05. This would give the sun a central density of about 1200 and a central temperature of just over 100 million degrees.

These figures are necessarily very vague, and the $\eta \propto T^{-1}$ law cannot at best represent the actual conditions of nature with much accuracy. But we can draw the general conclusion that the central densities and temperatures of main-sequence stars must be very substantially increased over the values calculated for them in Table II.; the increase will be specially marked in stars of great mass.

Stars which are well off the main sequence will probably have their generation of energy concentrated rather near to their centres, as a consequence of their heaviest atoms sinking towards the centre. This would require a reduction in the central temperatures and densities tabulated in Table II.

The general effect of these corrections is to space out central temperatures much more widely than they are spaced out in Table II. A central temperature of only 4 million degrees would not seem unreasonable for α Orionis, nor one of 400 million degrees for V Puppis; it would indeed be hard to assign any definite reason why this latter star should not have a central temperature of 4000 million degrees.

Summary.

10. The central densities and temperatures of certain stars have been calculated *de novo*, and prove to be much higher than has hitherto been supposed possible. The temperatures at the centres of the stars also vary far more widely than has been supposed; in particular those for stars on the main sequence show far more variation than was indicated by the calculations of Russell.

The results obtained are antagonistic to Russell's suggestion that the main sequence is formed of stars of a given central temperature, while rather favouring my more recent view of it as a sequence of configuration in which the atoms at the centre are stripped nearly or completely bare of electrons.

Adjusting the already calculated values so as to allow for all the conditions of the problem, the central density ρ_c and central temperatures T_c of a few selected stars would appear likely to be somewhat as follows :—

Star.	T_c.	ρ_c.
Plaskett's star	500 million degrees	v. great
V Puppis	300 ,,	5,000
Sirius	150 ,,	1,000
Sun	70 ,,	300
60 Kruger B	70 ,,	30,000
α Cent. B	15 ,,	10
α Orion.	4 ,,	0·1

It must be emphasised that most of these values, particularly those of the density, are very uncertain, amounting in many cases to little more than a personal impression of what values are most probable. The extreme limit of the main sequence has been supposed to be determined by the somewhat arbitrary condition $\eta \propto T^{-1}$; it has been assumed that Boyle's law holds up to the highest densities concerned, and that there is no appreciable nuclear disintegration at the highest temperatures concerned.

The Reflection Effect in Eclipsing Binaries.

By E. A. Milne, F.R.S.

In discussing the problem of radiative equilibrium near the boundary of a star it is usual to take the incident radiation as zero. In the case of close binaries the radiation from the one star at the surface of the other is appreciable, and it becomes important to discuss the radiative equilibrium of the outer layers of a star subject to external radiation.

The subject has recently been considered by Professor Eddington * in an illuminating paper. In the course of the investigation Professor Eddington suggested I might care to inquire into a problem not discussed by him, namely, the problem of the distribution of the "reflected" radiation in direction, or, in other words, of the law of darkening of the disc for the reflected radiation. The analysis required is identical with that for the problem of the radiative equilibrium of the atmosphere of a planet subject to insolation, on which I published a paper a few years ago under the title "Radiative Equilibrium : the Insolation of an Atmosphere." † The applications in that paper were, however, entirely meteorological. At Professor Eddington's suggestion I now communicate an account of the analysis in the form most suitable for application to the atmosphere of a star.

As Eddington pointed out, a star necessarily re-emits all the radiation falling on it, when it is in a steady state. In other words, for a star in a steady state the heat-albedo is unity. The effect of the radia-

* *M.N.*, **86**, 320, 1926. † *Phil. Mag.*, **44**, 871, 1922.

tion incident on the surface from outside is simply to maintain the surface layers at a temperature higher than they would otherwise have. When the state is steady, this extra temperature is just sufficient to enable the outer layers to radiate back to space an amount of energy equal to that incident on them, in addition to that which they would radiate in the absence of incident radiation. This is a consequence of the assumption of radiative equilibrium. The mathematical analysis which follows will in fact show that when the outer layers are in radiative equilibrium under the influence of external radiation, the net flux of external radiation transmitted to the far interior is zero at great depth. The whole of the incident radiation is necessarily re-radiated to space from the outer layers.*

This conclusion is independent of any considerations of the flux of energy from the far interior. This net flux we may state to be the same as in the absence of incident radiation. For the effect of the extra warming of the surface layers, or of the additional radiation-pressure due to the incident radiation, in altering the physical conditions in the far interior and hence in altering the rate of evolution of energy, must be negligible.

The incident radiation is absorbed in the outer layers and re-emitted. Following Eddington it is convenient to speak of this as " reflected " radiation, but it must not be confused with true reflection.

The principal problems with which we shall be concerned here are that of the increase of surface temperature due to the presence of the other star, and that of the determination of the law of distribution of the reflected radiation in direction.

We discuss in the first instance an idealised problem. Consider a semi-infinite mass of material with a plane boundary, stratified in parallel planes, and exposed to incident radiation on the boundary. We suppose the state to be a steady one, in strict radiative equilibrium. Let F be the net flux of energy per unit area, being the radiation generated in the far interior.

We now note that without loss of generality we can take $F = 0$. For solutions can be superposed. Suppose that

$$(\sigma/\pi)T^4 = B_1(\tau),$$

where T is the temperature at optical depth τ and σ is Stefan's constant, is a temperature distribution which gives a net flux F with the condition of zero incident radiation : and suppose further that

$$(\sigma/\pi)T^4 = B_2(\tau)$$

is a second temperature distribution which gives zero net flux with the incident radiation equal to its given value. Then all the equations (equations of transfer and of radiative equilibrium) are linear, and accordingly the temperature distribution

$$(\sigma/\pi)T^4 = B_1(\tau)+B_2(\tau)$$

* This proof of " heat-albedo in steady state = unity " is somewhat different from that given by Professor Eddington. I have profited by a discussion with Dr. Jeans on this point.

will give a net flux F with the incident radiation equal to its given value.

Further, the intensity of radiation I at any point may be written I_1+I_2, where I_1 is derived from the distribution B_1 and I_2 from B_2 by means of the equation of transfer in the usual way. The part I_1 will correspond to a net flux F, and in fact be identical with the known solution of the problem in the absence of external radiation: it will correspond approximately to a coefficient of darkening of 3/5. It remains to determine B_2 and I_2.

We now omit the suffixes, and determine a temperature distribu-

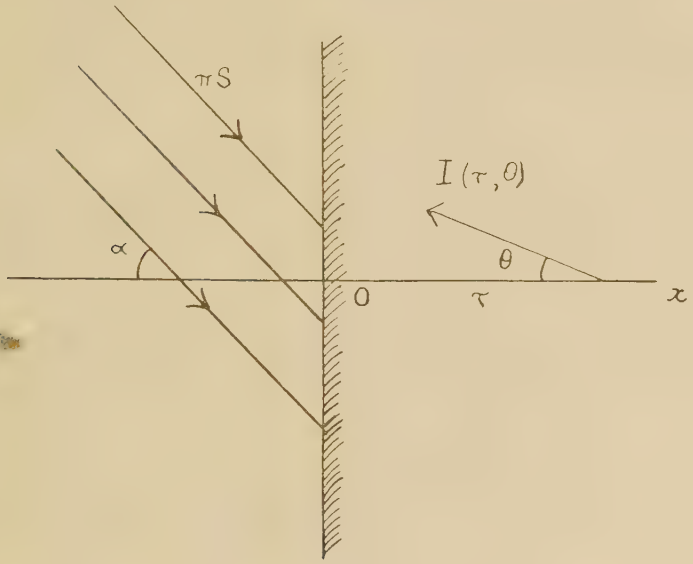

FIG. 1.

tion $B(\tau)$ corresponding to zero net flux with a given incident radiation. Let the incident radiation consist of a *parallel* beam, of flux πS per unit area normal to itself, incident in a direction making an angle α with the normal to the boundary (fig. 1). Then the flux incident on the boundary per unit area is $\pi S \cos\alpha$. This beam will be absorbed exponentially as it penetrates the material, and at a depth τ the flux contained in the parallel beam, per unit area perpendicular to the axis Ox is reduced to

$$\pi S e^{-\tau \sec\alpha} \cos\alpha,$$

where τ is the optical depth

$$\int_0^x k\rho dx$$

in the usual notation. Let $I(\tau, \theta)$ be the intensity of radiation at the

point τ in the direction θ, as derived from emissions from the material. It will be observed that we distinguish between the *beam* of flux

$$\pi S e^{-\tau \sec\alpha} \cos\alpha,$$

which though attenuated maintains its individuality at all depths, and the *intensity* $I(\tau, \theta)$ (a continuous function of θ) of the radiation excluding the beam.

The equation of transfer is as usual

$$\cos\theta \frac{dI(\tau, \theta)}{d\tau} = I(\tau, \theta) - B \quad . \quad . \quad . \quad (1)$$

where B is written for $B(\tau) = (\sigma/\pi)T^4$. The absorption of the incident beam in $d\tau$ at a depth τ is

$$\pi S e^{-\tau \sec\alpha} \cos\alpha \,.\, \sec\alpha \, d\tau$$

$$= \pi S e^{-\tau \sec\alpha} d\tau.$$

Hence the equation of radiative equilibrium expressing the equality of the absorbed and emitted energy is

$$\int I d\omega + \pi S e^{-\tau \sec\alpha} = 4\pi B \quad . \quad . \quad . \quad (2)$$

the integral being taken over the complete solid angle. Multiplying (1) by $d\omega$, integrating and using (2) we have

$$\frac{d}{d\tau}\int I \cos\theta \, d\omega = -\pi S e^{-\tau \sec\alpha},$$

which integrates with respect to τ in the form

$$\int I \cos\theta \, d\omega = \pi S \cos\alpha \, e^{-\tau \sec\alpha} \quad . \quad . \quad . \quad (3)$$

This equation represents the fact that the net flux is zero. For the left-hand side represents the outward flux, the right-hand side the inward flux.

Equations (1), (2), and their integral (3) can be discussed by the methods used in a previous paper.* In particular the following integral equations can be deduced for $B(\tau)$:—

$$B(\tau) = \tfrac{1}{4} S e^{-\tau \sec\alpha} + \tfrac{1}{2}\int_0^\infty B(t) Ei(|t-\tau|) dt \quad . \quad . \quad (4)$$

where $Ei(x)$ is the exponential-integral function : and

$$\phi'(\tau) = \tfrac{1}{4} S e^{-\tau \sec\alpha} + \int_0^\tau \frac{\phi(\tau+y)-\phi(\tau-y)}{2y} e^{-y} dy + \int_\tau^\infty \frac{\phi(\tau+y)}{2y} e^{-y} dy \quad (5)$$

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* *M.N.*, **81**, 361, 1921.

where

$$\phi(\tau) = \int_0^\tau B(\tau)d\tau, \qquad \phi'(\tau) = B(\tau).$$

From (5) it can be seen that $B'(\tau)$ has a singularity at $\tau = 0$, becoming infinite like $\log \tau$.

To obtain approximate solutions it is better to use the original equations, and to approximate by reducing them to the "equations of linear flow" in the usual way. Using now I and I′ to denote the mean outward and inward intensities, defined by

$$I = \int_0^{\frac{1}{2}\pi} I(\tau, \theta) \sin\theta \, d\theta, \qquad I' = \int_{\frac{1}{2}\pi}^{\pi} I(\tau, \theta) \sin\theta \, d\theta,$$

equation (1) splits up in the usual way into

$$\tfrac{1}{2}\frac{dI}{d\tau} = I - B, \qquad \tfrac{1}{2}\frac{dI'}{d\tau} = B - I' \quad . \quad . \quad (6), (7)$$

whilst (2) and (3) reduce to

$$I + I' + \tfrac{1}{2}Se^{-\tau \sec\alpha} = 2B \quad . \quad . \quad . \quad . \quad (8)$$

$$I - I' = S \cos\alpha \, e^{-\tau \sec\alpha} \quad . \quad . \quad . \quad . \quad (9)$$

It should be noted that (9) is merely an integral of (6) and (7) when (8) is taken into account.

We have from (8) and (9)

$$I = B + \tfrac{1}{2}Se^{-\tau \sec\alpha}(\cos\alpha - \tfrac{1}{2}) \quad . \quad . \quad . \quad (10)$$

$$I' = B - \tfrac{1}{2}Se^{-\tau \sec\alpha}(\cos\alpha + \tfrac{1}{2}) \quad . \quad . \quad . \quad (11)$$

and inserting for I in (6) and integrating, we find

$$B(\tau) = a - S(\cos^2\alpha - \tfrac{1}{4})e^{-\tau \sec\alpha} \quad . \quad . \quad . \quad (12)$$

where a is a constant. The boundary condition is $I'(0) = 0$. This determines a, and we have as our approximation to B

$$B(\tau) = S(\cos\alpha + \tfrac{1}{2})[\cos\alpha - (\cos\alpha - \tfrac{1}{2})e^{-\tau \sec\alpha}] \quad . \quad . \quad (13)$$

This, it must be remembered, must be superposed on the ordinary temperature distribution. Considered by itself, it gives a temperature distribution with a definite boundary temperature and a definite limiting temperature in the far interior. If the incident flux πS is equated to $\sigma T_1'^4$, so that T_1' is the effective temperature of the incident radiation, then writing

$$B(0) = (\sigma/\pi)T_0'^4, \qquad B(\infty) = (\sigma/\pi)T_\infty'^4,$$

we find

$$T_0'^4 = \tfrac{1}{2}T_1'^4(\tfrac{1}{2} + \cos\alpha)$$

$$T_\infty'^4 = T_1'^4 \cos\alpha \, (\tfrac{1}{2} + \cos\alpha).$$

If T_1 is the effective temperature of the material in the absence of external radiation, the true boundary temperature T_0 is given by *

$$T_0^4 = \tfrac{7}{16}T_1^4 + \tfrac{1}{2}T_1'^4(\tfrac{1}{2}+\cos\alpha).$$

We can now use (13) to determine the emergent radiation $I(0, \theta)$ as a function of θ by introducing for B in (1) and solving in the usual way. We find

$$I(0, \theta) = \int_0^\infty B(\tau)e^{-\tau\sec\theta}\sec\theta\, d\tau$$

$$= S\frac{\cos\alpha\,(\cos\alpha+\frac{1}{2})(\cos\theta+\frac{1}{2})}{\cos\theta+\cos\alpha} \quad . \quad . \quad . \quad (14)$$

This gives the "law of darkening" of the reflected radiation. It is quite different from the law of darkening of the ordinary radiation.

We will now examine the errors of this approximation. The outward flux given by (14) is

$$2\pi\int_0^{\frac{1}{2}\pi} I(0, \theta)\cos\theta\sin\theta\, d\theta = \pi S\cos\alpha[1-\cos\alpha\,(2\cos\alpha-1)\{(\cos\alpha+\tfrac{1}{2})\log(1+\sec\alpha)-1\}],$$

whilst the inward flux is $\pi S\cos\alpha$. The two should be equal. The following table shows the values of the proportional error, namely,

$$\cos\alpha(2\cos\alpha-1)[(\cos\alpha+\tfrac{1}{2})\log(1+\sec\alpha)-1].$$

$\cos\alpha$.	Proportional Error in Flux.
1	+0·039
0·8	+0·026
0·5	0·000
0·3	−0·021
0·2	−0·031
0·1	−0·035
0·0	0·000

It will be seen that the error is always less than 4 per cent.

* We here use for the boundary temperature in the absence of incident radiation the relation $T_0^4 = \frac{7}{16}T_1^4$ instead of the more usual Schwarzschild value $T_0^4 = \frac{1}{2}T_1^4$. The former may be shown to be appreciably more accurate. It is easily derived as follows. It is known that radiative equilibrium corresponds very closely to a coefficient of darkening 3/5, and hence the emergent distribution is of the form

$$I(\theta) = C(\tfrac{2}{5}+\tfrac{3}{5}\cos\theta).$$

If the net flux is πF, we find

$$\pi F = 2\pi\int_0^{\frac{1}{2}\pi} I(\theta)\cos\theta\,.\,\sin\theta\,.\,d\theta\,. = 4\pi C/5,$$

fixing C. The temperature of the boundary layer is given by the condition of radiative equilibrium

$$4\pi B(0) = 2\pi\int_0^{\frac{1}{2}\pi} I(\theta)\sin\theta\, d\theta = 7\pi C/5 = 7\pi F/4.$$

Putting

$$\pi F = \sigma T_1^4, \qquad B(0) = (\sigma/\pi)T_0^4,$$

we find

$$T_0^4 = \tfrac{7}{16}T_1^4.$$

Another test is to work out the extent to which the condition of radiative equilibrium is satisfied at the boundary. Per unit optical mass, the emission is $4\pi B(0)$, or

$$2\pi S\,(\cos\alpha+\tfrac{1}{2}),$$

whilst the absorption is

$$2\pi\int_0^{\frac{1}{2}\pi} I(0,\,\theta)\sin\theta\,d\theta+\pi S$$
$$= 2\pi S[(\cos\alpha+\tfrac{1}{2})+\cos\alpha\,(\cos\alpha-\tfrac{1}{2})\{1-(\cos\alpha+\tfrac{1}{2})\log(1+\sec\alpha)\}].$$

We see that the error vanishes for $\cos\alpha = 0$ and $\cos\alpha = \frac{1}{2}$. For $\cos\alpha = 1$, the absorption is $2{\cdot}961\pi S$, the emission $3\pi S$, an error of 1·3 per cent.

We proceed to consider a second approximation. We see that according to (13), $B(\tau)$ is of the form

$$B(\tau) = a-be^{-\tau\sec\alpha} \quad . \quad . \quad . \quad . \quad (15)$$

and it is clear that one way of obtaining a better approximation is to assume this form for $B(\tau)$ and then determine a and b, so that the flux condition and the radiative equilibrium condition are exactly satisfied at the boundary. (Reflection, *i.e.* absorption and re-emission, occurs chiefly in the superficial layers, so that it is natural to choose our constants so as to get the best fit at the boundary. In the far interior the flux given by (15) automatically tends to zero asymptotically, since for τ large the temperature distribution is practically the isothermal one $T = T_\infty$.)

The emergent radiation corresponding to (15) is

$$I(0,\,\theta) = a-b\frac{\cos\alpha}{\cos\alpha+\cos\theta} \quad . \quad . \quad . \quad (16)$$

Inserting this in the flux-equation (3) and in the condition of radiative equilibrium (2) with $\tau = 0$ we find, respectively,

$$a-2b\,\cos\alpha[1-\cos\alpha\log(1+\sec\alpha)] = S\cos\alpha,$$
$$a-b[2-\cos\alpha\log(1+\sec\alpha)] = \tfrac{1}{2}S,$$

whence

$$\frac{a}{\cos\alpha} = \frac{b}{\cos\alpha-\frac{1}{2}} = \frac{\frac{1}{2}S}{1-\cos\alpha+\cos\alpha\,(\cos\alpha-\frac{1}{2})\log(1+\sec\alpha)}.$$

For $\cos\alpha = \frac{1}{2}$ and $\cos\alpha = 0$, the values of a and b are the same as on the first approximation (13). For $\cos\alpha = 1$, we find $a = S/\log 2 = 1{\cdot}443S$ instead of $a = 1{\cdot}5S$, and $b = S/2\log 2 = 0{\cdot}721S$ instead of $b = 0{\cdot}75S$. The changes are trifling. The emergent radiation is now, from (16),

$$I(0,\,\theta) = a\,\frac{\cos\theta+\frac{1}{2}}{\cos\theta+\cos\alpha} \quad . \quad . \quad . \quad . \quad (17)$$

and so the law of darkening is unaltered.

The general conclusion is that the temperature distribution (13) and the law of darkening (14) are good approximations, sufficiently accurate for practical purposes.

The distribution of temperature given by (13) is of interest. When $\cos\alpha = \frac{1}{2}$, the distribution is isothermal, being given by

$$B(\tau) = (\sigma/\pi)T^4 = S\cos\alpha\,(\cos\alpha+\tfrac{1}{2}).$$

When $\cos\alpha > \frac{1}{2}$, the temperature increases from the surface inwards: when $\cos\alpha < \frac{1}{2}$, it decreases. These results apply to the case where there is no flux from the far interior. When we superpose the effect of the latter, we see that in the far interior the temperature at any point is raised by the external radiation from T″ to T, where

$$T^4 = T''^4 + T_\infty'^4$$

and $T_\infty'^4$ has the value given above.

We now apply the results to a close binary. Consider first the reflection at full phase, when as seen from the earth the centre S_2 of the secondary is behind the centre S_1 of the primary (see fig. 2). As a first approximation we may consider the light from the primary as constituting a parallel beam. The angle of incidence α at any point P of the secondary is then approximately equal to the angle of emergence θ of the light radiated to the earth from P. If L_1 is the luminosity of the primary, a the distance between the centres, the flux πS at P is

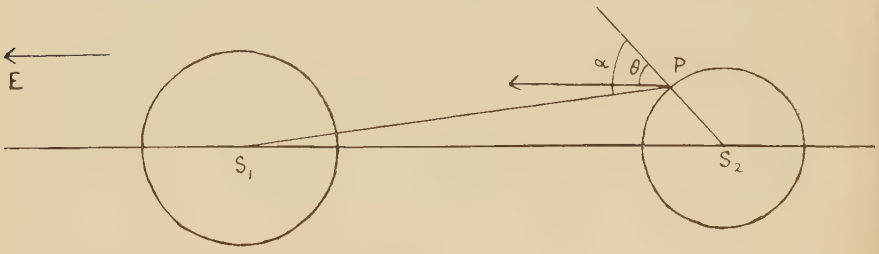


Fig. 2.

approximately $L_1/4\pi a^2$, and the intensity of light reflected towards the earth is

$$I(0, \theta) = \frac{L_1}{4\pi a^2} \cdot \frac{1}{\pi}\left[\frac{\cos\alpha\,(\cos\alpha+\frac{1}{2})(\cos\theta+\frac{1}{2})}{\cos\alpha+\cos\theta}\right]_{\alpha=\theta}$$

$$= \frac{L_1}{4\pi a^2}\,\frac{(\cos\theta+\frac{1}{2})^2}{2\pi} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The law of darkening of the disc of the secondary for reflected light, at full phase, is accordingly

$$\frac{I(0, \theta)}{I(0, 0)} = \left(\frac{1+2\cos\theta}{3}\right)^2 \quad . \quad . \quad . \quad . \quad (19)$$

For example, at the limb the ratio $I(0, \frac{1}{2}\pi)/I(0, 0)$ is 1/9. The reflected light thus shows enormous darkening at the limb compared with the ordinary light.

To find the ratio of reflected light to ordinary light at full phase we proceed as follows. The total light reflected to the earth inside a small solid angle $d\Omega$ from an element of area $d\sigma_2$ of the secondary is

$$\frac{L_1}{4\pi a^2} \cdot \frac{1}{2\pi}(\cos\theta+\tfrac{1}{2})^2 \cos\theta \, . \, d\Omega d\sigma_2$$

by definition of the " intensity of radiation," where

$$d\sigma_2 = r_2^2 \sin\theta \,.\, d\theta \,.\, d\phi,$$

r_2 being the radius of the secondary. The ordinary radiation from the whole disc of the secondary is $L_2 d\Omega/4\pi$. Hence the ratio s for the whole disc is approximately

$$s = \frac{\text{reflected light}}{\text{ordinary light}}$$

$$= \frac{L_1}{L_2}\frac{r_2^2}{a^2}\int_0^{\frac{1}{2}\pi} (\cos\theta+\tfrac{1}{2})^2 \cos\theta \sin\theta \, d\theta$$

$$= \frac{17}{24}\frac{L_1}{L_2}\frac{r_2^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)^*$$

The observed luminosity is of course $L_2(1+s)$. This result has already been given by Eddington (*loc. cit.*, p. 322, equation (5)). As he there mentions, it so happens that the same result is obtained if we (erroneously) assume the law of darkening for the reflected light to be the same as that for the ordinary light, namely, a simple cosine law with coefficient of darkening 3/5.

The increase in T_0^4 at P due to the incident radiation, which we have called $T_0'^4$, is given by

$$\sigma T_0'^4 = \tfrac{1}{2}\pi S \,(\cos\alpha+\tfrac{1}{2})_{\alpha=\theta}$$

$$= \tfrac{1}{2}\,(\cos\theta+\tfrac{1}{2})\frac{L_1}{4\pi a^2} \,. \quad . \quad . \quad . \quad (21)$$

The surface temperature of the secondary in the absence of incident radiation is T_0, given by

$$\sigma T_0^4 = \frac{7}{16}\frac{L_2}{4\pi r_2^2}.$$

Hence

$$\frac{T_0'^4}{T_0^4} = \frac{8}{7}\cdot\frac{L_1}{L_2}\cdot\frac{r_2^2}{a^2}(\cos\theta+\tfrac{1}{2})$$

$$= \frac{8}{7}\cdot\frac{24}{17}\cdot(\cos\theta+\tfrac{1}{2})s \quad . \quad . \quad . \quad (22)$$

The change in temperature thus varies from point to point of the disc.

For discussing the change in spectral type due to reflection we require the average value of the surface temperature. The fair way to average appears to be to weight according to the amount of radiation from each element of the disc, ordinary plus reflected. The weighting factor is thus the expression

$$\left[\frac{L_2}{4\pi r_2^2}\cdot\frac{2+3\cos\theta}{4\pi}+\frac{L_1}{4\pi a^2}\cdot\frac{(\cos\theta+\frac{1}{2})^2}{2\pi}\right]\cos\theta$$

* For simplicity the upper limit of integration has been taken as $\frac{1}{2}\pi$ instead of $\cos^{-1}(r_2/a)$. The necessary correction has been given by Eddington.

divided by its integral over the disc, namely,

$$\frac{L_2}{4\pi r_2^2}(1+s).$$

We find then with this weighting factor

$$\overline{\cos\theta} = \frac{17}{24}\cdot\frac{1+\frac{8}{15}(\frac{24}{17})^2 s}{1+s} = \frac{17}{24}\cdot\frac{1+1{\cdot}06s}{1+s}.$$

We can call this simply 17/24. Then putting this mean value in (22) we have

$$\overline{T_0'^4}/T_0^4 = \tfrac{8}{7}\cdot\tfrac{29}{17}s = 1{\cdot}95s.$$

The fourth power of the surface temperature as influencing the spectral composition of the light is thus $T_0^4(1+1{\cdot}95s)$. The increase in mean

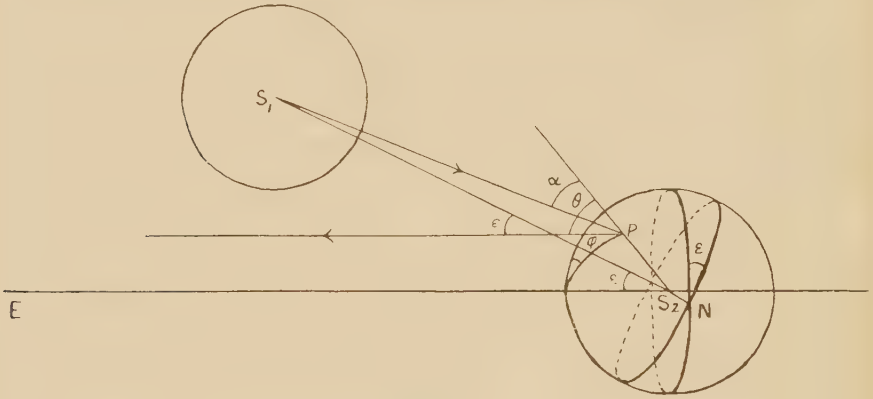


Fig. 3.

surface temperature thus exceeds the increase in effective temperature as judged by the increase in surface brightness. Eddington has already pointed out that the observations suggest a strong tendency in this direction, namely, a tendency for the spectrum of the secondary to be "earlier" than that inferred from its surface brightness. Two effects of a physical character were adduced by Eddington to account for this. Our analysis supplies a third.

In Eddington's notation, s is equal to $L_r/(1-L_1-L_r)$, L_r having its observed value. For the twelve stars listed by Eddington, the following table shows the value of $(1+2s)^{\frac{1}{4}}$, which is approximately the factor by which the surface temperature is increased by the presence of the primary.

No. of Star.	$(1+2s)^{\frac{1}{4}}$.	No. of Star.	$(1+2s)^{\frac{1}{4}}$.
1	1·08	7	1·09
2	1·15	8	1·26
3	1·31	9	1·23
4	1·46	10	1·12
5	1·08	11	2·14
6	1·27	12	1·06

When the phase is not full, the algebra is more complicated, because only a portion of the hemisphere of S_2 illuminated by S_1 is visible from the earth E. Let the phase be an angle ϵ from full. Fig 4 shows the projections of the various directions of fig. 3, projected on to the surface of S_2.* In fig. 4 the angle $PES_1 = \phi$ is the angle between the planes PS_2E and S_1S_2E of fig. 3, and thus measures the azimuth of P on the disc measured from the plane S_1S_2E. In terms of ϕ and θ the angle α is given by

$$\cos\alpha = \cos\theta\cos\epsilon + \sin\theta\sin\epsilon\cos\phi.$$

To illustrate the law of darkening of the reflected light over the disc

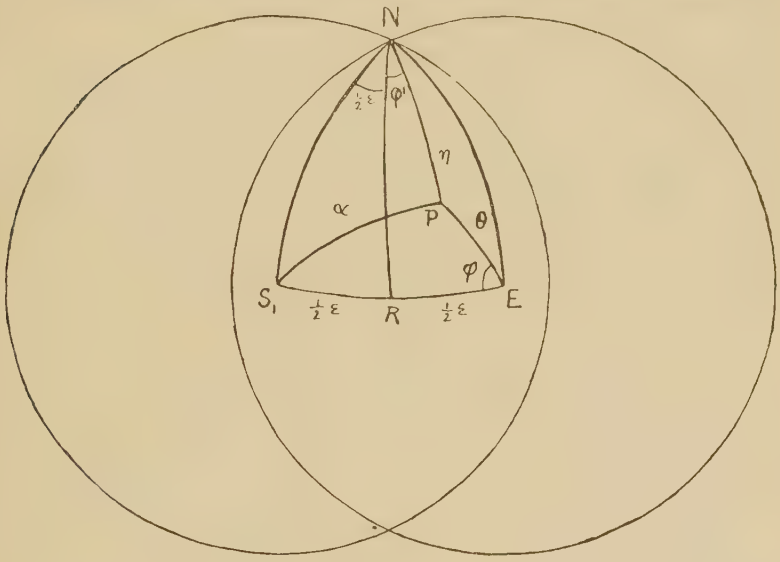


Fig. 4.

we will consider briefly the central azimuth $\phi = 0$, along which $\alpha = \theta - \epsilon$. Along this,

$$I(0, \theta) = \frac{L_1 r_2^2}{4\pi a^2} \cdot \frac{1}{\pi} \cdot \frac{\cos(\theta-\epsilon)\{\cos(\theta-\epsilon)+\frac{1}{2}\}(\cos\theta+\frac{1}{2})}{\cos(\theta-\epsilon)+\cos\theta},$$

where $-(\frac{1}{2}\pi-\epsilon)<\theta<\frac{1}{2}\pi$. For example, the ratios

$$I(0, \tfrac{1}{2}\pi) : I(0, \epsilon) : I(0, 0) : I(0, -(\tfrac{1}{2}\pi-\epsilon))$$

$$= \tfrac{1}{2}(\sin\epsilon+\tfrac{1}{2}) : \frac{\frac{3}{2}(\cos\epsilon+\frac{1}{2})}{\cos\epsilon+1} : \frac{\frac{3}{2}\cos\epsilon(\cos\epsilon+\frac{1}{2})}{\cos\epsilon+1} : 0$$

show the relative intensities at four points.

* We assume as before that the radiation from S_1 constitutes an approximately parallel beam.

The total light reflected to the earth inside a small solid angle $d\Omega$ is now

$$d\Omega \frac{L_1 r_2^2}{4\pi a^2} \cdot \frac{1}{\pi} \int \frac{\cos\alpha\,(\cos\alpha+\frac{1}{2})(\cos\theta+\frac{1}{2})}{\cos\alpha+\cos\theta} \cos\theta\, d\omega$$

the integral being extended to the lune enclosed between the two great circles with poles S_1 and E. To evaluate this integral we take N, a point of intersection of the great circles, as pole and express the position of P by means of polar co-ordinates

$$\eta = \mathrm{NP}, \qquad \phi' = \mathrm{P\hat{N}R},$$

where R is the bisector of S_1E. We have then

$$\begin{aligned} d\omega &= \sin\eta\, d\eta d\phi', \\ \cos\alpha &= \sin\eta \cos(\tfrac{1}{2}\epsilon+\phi'), \\ \cos\theta &= \sin\eta \cos(\tfrac{1}{2}\epsilon-\phi'). \end{aligned}$$

The limits for η are 0 and π, for ϕ' are $-(\frac{1}{2}\pi-\frac{1}{2}\epsilon)$ and $\frac{1}{2}\pi-\frac{1}{2}\epsilon$. The ordinary light is $L_2 d\Omega/4\pi$, as before. After some troublesome algebra we find that the ratio $s(\epsilon)$ of reflected light to ordinary light at phase angle ϵ is given by

$$s(\epsilon) = \frac{L_1}{L_2} \frac{r_2^2}{a^2} f(\epsilon) \quad . \quad . \quad . \quad . \quad (24)$$

where

$$f(\epsilon) = \frac{(1-\cos\epsilon)(1-3\cos\epsilon)}{32\cos\frac{1}{2}\epsilon} \log \frac{1+\cos\frac{1}{2}\epsilon}{\sin\frac{1}{2}\epsilon} + \frac{1+5\cos\epsilon}{16} + \frac{\sin\epsilon+(\pi-\epsilon)\cos\epsilon}{3\pi} \quad (25)$$

Particular values are

$$f(0) = \tfrac{17}{24} = 0{\cdot}708\,; \qquad f(\tfrac{1}{2}\pi) = 0{\cdot}2075\,; \qquad f(\pi) = 0.$$

Eddington found that if the darkening law for the reflected radiation is neglected, so that the reflected radiation is supposed to be distributed uniformly in direction, then

$$f(\epsilon) \propto \frac{\sin\epsilon+(\pi-\epsilon)\cos\epsilon}{\pi}.$$

It will be found that the correct function allowing for darkening, namely (25), is closely the same as

$$f(\epsilon) = \frac{17}{24} \cdot \frac{\sin\epsilon+(\pi-\epsilon)\cos\epsilon}{\pi}.$$

For example, the latter agrees with (25) at $\epsilon = 0$ and $\epsilon = \pi$, and at $\epsilon = \frac{1}{2}\pi$ gives $f(\frac{1}{2}\pi) = 0{\cdot}226$ instead of $0{\cdot}2075$.

The albedo in a given direction and the law of darkening of the disc could be worked out if necessary, taking into account the finite solid angle subtended by the primary at the secondary, but the results would be very complicated. Our results are correct to the first term in r_2/a.

Summary.

1. If a parallel beam of flux πS per unit cross-section is incident at angle α on the plane surface of a semi-infinite mass of material in radiative equilibrium, the radiation emerges as a diffuse beam whose intensity $I(o, \theta)$ in a direction θ with the normal is given by

$$I(o, \theta) = S \frac{\cos\alpha\ (\cos\alpha+\frac{1}{2})(\cos\theta+\frac{1}{2})}{\cos\alpha+\cos\theta}.$$

The temperature T at optical depth τ is given by

$$\sigma T^4 = \pi S\ (\cos\alpha+\tfrac{1}{2})[\cos\alpha-(\cos\alpha-\tfrac{1}{2})e^{-\tau \sec\alpha}].$$

There is a definite boundary temperature and a definite limiting temperature in the far interior.

2. This solution for T^4 may be superposed on any other solution corresponding to radiation generated in the far interior and transmitted to the outer layers. In particular it may be superposed on the standard solution for radiative equilibrium corresponding to zero incident radiation.

3. The above results for an idealised model may be applied to the problem of the reflection of light in close binaries. When the phase is full the distribution of reflected light over the disc of the secondary is given by

$$I(o, \theta) = \frac{L_1}{4\pi a^2} \frac{(\cos\theta+\frac{1}{2})^2}{2\pi},$$

where L_1 is the luminosity of the primary and a is the distance between the centres of the stars.

4. The ratio s of reflected light to ordinary light at full phase is given approximately by

$$s = \frac{17}{24} \frac{L_1}{L_2} \frac{r_2^2}{a^2},$$

where L_2, r_2 are the luminosity and radius of the secondary. This result has been given by Eddington, using an approximate theory in which the distribution of reflected light was supposed to be the same as that of the ordinary light.

5. The surface temperature of the secondary is such that T_0^4 is increased by reflection by the factor $(1+2s)$, averaged over the disc according to the intensity of total radiation. The surface temperature thus increases more than would be inferred from the increase in surface brightness.

6. Formulæ are obtained from which the reflected light and its distribution may be calculated for phases other than the full.

(*Note.*—In the foregoing the word " light " is used, for brevity, for " total radiation of all wave-lengths.")

The University, Manchester :
1926 November.

On the Chromospheric Currents above a Sunspot. By S. R. Pike.

(*Communicated by Professor E. A. Milne, F.R.S.*)

It has been shown by Milne * that the atoms in the solar chromosphere are mainly supported by radiation-pressure ; and for the simplest type of atom such as Ca+ he has derived the equation

$$\left(\alpha\frac{q_1}{q_2}\right)\tau = \frac{r}{4mgc}\cdot\frac{h\nu}{e^{h\nu/k\mathrm{T}}-1} \quad . \quad . \quad . \quad (1)$$

where τ is the " life " of the excited atom, r the residual intensity in the absorption line ν (for Ca+, the mean of the H, K lines), and q_1, q_2 are the " weights " of the normal and excited states. The factor α is introduced to allow for the presence of the metastable $3d$ level between the $1s$ and $2p$ levels which are responsible for the resonance line. The true value of τ is then found by taking $r = \frac{1}{9}$, $\mathrm{T} = 6000°$ A. (for the sun), $\alpha = 0{\cdot}72$, $\frac{q_1}{q_2} = \frac{1}{3}$ which gives

$$\tau = 1{\cdot}3\times10^{-8} \text{ sec.}$$

but for our present purpose we may consider the factor $\left(\frac{\alpha q_1}{q_2}\right)$ to be implicit in τ, for which we then take the value as

$$\tau = 0{\cdot}6\times10^{-8} \text{ sec.}$$

We can rewrite (1) in the form

$$r = \frac{4mgc\tau}{h\nu}\cdot[e^{h\nu/k\mathrm{T}}-1] \quad . \quad . \quad . \quad . \quad (2)$$

and, as Milne pointed out, the condition $r \not> 1$ sets a lower limit to the temperature at which it is possible for a star to possess a chromosphere at all. In general, if a star sinks to a temperature $\mathrm{T}' < \mathrm{T}$, the value of r will be adjusted to r' where

$$r' = \frac{e^{h\nu/k\mathrm{T}'}-1}{e^{h\nu/k\mathrm{T}'}-1}\cdot r \quad . \quad . \quad . \quad . \quad (3)$$

which, since $r' > r$, can only be achieved by diminishing the total number of Ca+ atoms in the atmosphere.

We shall show that if any area on the surface of a star at temperature T is reduced to a temperature T′, then the same condition applies to the chromosphere immediately above such an area, and the value of r must rise to the same value r' as in the case of a star wholly at T′.

Consider first for simplicity a single calcium atom A at a height h above the centre of a circular area of radius r in which the temperature has fallen from T to T′.

Then the solid angle subtended by this area at A is

$$\omega = 4\pi\sin^2\tfrac{1}{2}\phi \quad \text{where} \quad \phi = \cos^{-1}h/r.$$

and the remainder of the disc thus subtends the angle $2\pi-\omega = \omega'$ say.

* Milne, *M.N.*, **84**, 354 ; **85**, 111 ; **86**, 8.

Following Milne's notation, and putting

I_ν = intensity of isotropic radiation from disc

I_ν' = intensity of isotropic radiation from darkened area

we have

$$\frac{\tau'}{\tau} = \frac{1}{B_{1\to 2}(\omega I_\nu' + \omega' I_\nu)}$$

and

$$\frac{1}{\tau} = A_{2\to 1} = \frac{2h\nu^3}{c^2} B_{1\to 2}.$$

Therefore

$$\frac{\tau'}{\tau} = \frac{2h\nu^3}{c^2(\omega I_\nu' + \omega' I_\nu)} \quad . \quad . \quad . \quad . \quad (4)$$

Again, the vertical impulse * received from the absorption of a quantum in any direction θ is $\frac{h\nu}{c}\cos\theta$, and thus, neglecting τ compared with τ',

$$mg\tau' = \frac{\frac{h\nu}{c}[\cos\theta]_0^{\phi} \cdot \omega I_\nu' + \frac{h\nu}{c}[\cos\theta]_{\phi}^{\pi/2} \cdot \omega' I_\nu}{\omega I_\nu' + \omega' I_\nu} \quad . \quad . \quad (5)$$

where $[\cos\theta]_0^{\phi}$ = average value of $\cos\theta$ over the solid angle ω, *i.e.* $\cos^2 \frac{1}{2}\phi$ and similarly $[\cos\theta]_{\phi}^{\pi/2} = \frac{1}{2}\cos\phi$.

Combining (4) and (5),

$$\tau = [I_\nu' \sin^2\phi + I_\nu \cos^2\phi] \frac{c}{2\nu^2 mg},$$

and if we take

$$I_\nu' = r' \cdot \frac{2h\nu^3}{e^{h\nu/kT} - 1}$$

and remember that

$$\tau = \frac{e}{8\nu^2 mg} \cdot I_\nu$$

we obtain the equation

$$\tau = \frac{h\nu}{4mgc} \cdot \frac{r'}{e^{h\nu/kT} - 1} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This proves our statement for the case of a circular area, since (6) is identical with (2); and plainly it is easy to generalize it to include areas of any shape whatever. For the region outside the area in question will contribute the same fraction of the total support required whether it has the limiting temperature and $r = 1$, or whether it is hotter and has the appropriate smaller value of r. Thus the equilibrium is conditioned solely by the temperature of the darkened area, not by its size (or shape).

Now let us apply this equation to the particular circumstances of

* A similar calculation has recently been made by R. K. Sur, *Astrophys. Journ.*, **63**, 2, p. 111, for the case of an area brighter than its surroundings.

a sunspot. The umbra is known to give a spectrum of type Ko, so that its temperature can hardly exceed 4500° A.; thus, inserting numerical values, we find that the equilibrium value of r' is

$$r' = 0{\cdot}83.$$

Physically speaking, this means that the Ca+ atoms above the umbra of a sunspot will descend until the absorption in the H and K lines has reached the very low value necessary to make r nearly unity. In other words, the partial pressure of the calcium vapour over the spot will fall almost to zero; and the atoms in the surrounding chromosphere will consequently diffuse inwards with a velocity of the same order as that of their thermal agitation (say 2 km./sec.). The process will, of course, continue so long as the spot persists; it will act as a sort of sink, sucking in the atoms at the high levels. Presumably they will be thrown out again somewhere in the reversing layer; for as soon as a region is reached in which gas-pressure predominates over radiation-pressure, the atoms will accumulate until a tendency to outward diffusion begins to set in.

This conclusion is confirmed by the observations of Evershed * and St. John † on the high level currents in the sun. The motion of matter over a sunspot depends in a very marked way upon its height above the surface. At low levels, all the constituents of the atmosphere appear to be moving radially outwards from the axis of the spot in horizontal planes: and this motion is interpreted as being due to the upper parts of the actual spot vortex, in which the tangential velocities have already died out at a level too deep for observation. The outflow velocities decrease regularly with height, and at about 1500 km. (a point which also marks the usually accepted division between reversing layer and chromosphere) they die out altogether. After this, velocities of inflow begin to make their appearance in the high level chromospheric gases, *e.g.* Na and Ti at 5000 km., H at 8000, and Ca+ at 14,000; these velocities increase with height, till at 14,000 km. a value of 3·8 km./sec. is attained. Tangential velocities also make their appearance; and St. John supposes this upper vortex to be due simply to a swirling inflow of gases, having only a very loose connexion with the vortical motion in the spot itself. He also quotes an observation by Respighi ‡ to the effect that over a spot the chromosphere is generally very low and sometimes missing altogether.

All this fits in very well with the consequences we should expect from the failure of the radiation from a sunspot to support a calcium chromosphere; and although we cannot suppose that the general outflow observed in the reversing layer is due to the precipitation of elements from above (since so many are involved which do not appear in the chromosphere at all), it is nevertheless very easy to see how those elements which do fall in from the chromosphere can be restored to it. For when they reach a low level, they will begin to flow outwards for two reasons:—

* Evershed, *Kodaikanal Obs. Bull.*, 15.
† St. John, *Astrophys. Journ.*, **37**, 5, p. 322; **38**, 4, p. 341; **60**, 1, p. 32.
‡ *Am. Journ. Sci.*, **1** (1871).

(*a*) Their own increasing partial pressure will promote diffusion.
(*b*) They will be entrained by the other outflowing vapours, coming from the actual spot vortex.

As soon as a normal region of the photosphere is reached, the radiation-pressure will again raise them into the chromosphere to compensate for those atoms which are being lost to it over the umbra. Evidence * of such an upward flow has been obtained by St. John, although curiously enough the vapour appears to be stationary over the flocculi and the penumbra.

When we turn to the consideration of other elements present in the chromosphere which are not of the simple type of Ca+, Sr+, etc., it is impossible to apply Milne's method in its original form. The most important of these elements is hydrogen; and for this substance (together with several others) an interesting alternative method has been developed by M. C. Johnson.† This method depends upon the assumption (*prima facie* a very reasonable one) that the monochromatic mass absorption coefficient k_ν is of the order 10^8 for the important lines of an atom.

The upward acceleration of an atom due to the absorption of vertical radiation is then

$$R = \frac{k_\nu}{c} \cdot \tau_1 \cdot E_\nu \Delta_\nu,$$

where τ_1 is the fraction of its life during which an atom is capable of absorbing radiation of frequency ν, and $E\nu\Delta\nu$ is the energy available for absorption in that region of the spectrum.

Now at a temperature of 6000° A. and a pressure of 10^{-4} atmosphere (*i.e.* in the reversing layer) hydrogen has not yet begun to be ionised, so that every atom is in a state to absorb the Lyman, Balmer, or one of the higher series. Most of the atoms will be in the 1-quantum state, and so ready for Lyman absorption; whilst the fraction f_r in the r quantum state will be given by the usual Fowler-Milne equation. Thus for $r=2$, we find $f_r = 8\times10^{-9}$ at 6000° A.

Since the energies available at the first Balmer line and at the Balmer limit are only 5×10^5 times that available at the Lyman limit, and 3×10^3 times that available at the first Lyman line, it is evident that the greater part of the acceleration must be provided by Lyman absorptions. Taking $\Delta\nu$ to correspond to 2A°, Johnson finds that in a GO giant of mass 10 the radiation-pressure on a hydrogen atom is only $\frac{1}{6}{}_0$ of gravity. On the sun, gravity is some 10^3 times larger, and we should thus expect the effect of radiation-pressure to be quite negligible. On Johnson's theory, therefore, the hydrogen in the sun should be in purely gravitational equilibrium.

Such a conclusion is, of course, quite at variance with the known facts, for H_α has been traced up to 11,000 or 12,000 km., and all the lines from H_β to H_ν are seen at or above 6000 km. It has been shown by Milne that under such circumstances all but a very small fraction of the total weight of the atoms must be borne by radiation-pressure,

* *Loc. cit.*, **37**, 5, p. 349.
† *M.N.*, **85**, 8, p. 813; **86**, 5, p. 300.

as opposed to gas-pressure; and although it is true that this result has only been definitely proved for atoms of the Ca+ type, there is no reason to doubt that the same general conclusion must hold good for atoms which can be supported by any form of absorption whatever.

Again, we know that the partial electron pressure at a height of 14,000 km. is of the order 10^{-13} atmosphere, so that at 11,000 km. it is probably between 10^{-11} and 10^{-12} atmosphere. Now the ionisation of hydrogen at 6000° is completed between 10^{-10} and 10^{-11} atmosphere (we choose 6000° because the mechanism of ionisation is almost entirely photoelectric at such pressures); and hence we see that a hydrogen atom can ascend to heights at which it is bound to spend the greater part of its life in the ionised state, *i.e.* incapable of absorption. The absorptive power of the neutral atom must accordingly be increased by a further large factor.

In general, we may conclude with some certainty from the observed distribution of hydrogen in the chromosphere that Johnson's method of calculating the radiation-pressure gives a value which is too small by a factor of at least 10^5.

There are several reasons which might account for this :—

1. The assumed value of r ($= \frac{1}{3}$) may be too low.
2. The value $k_\nu = 10^8$ may be too low.
3. There may be many more lines taking part in the absorption than has been supposed.
4. The interval $\Delta\nu$ may be too small.
5. The 2-quantum * orbit of hydrogen may be metastable.

Of these possibilities (1) and (4) may at once be ruled out as inadequate. As regards (2), a value as high as 10^{10} for k_ν might perhaps be admissible; and taken together with (3), the discrepancy could be reduced to a factor of 50 or so. For a great many hydrogen lines are certainly observed in the sun: Mitchell recorded some 34 in the flash spectrum, and it is not impossible that there are an equal number of Lyman lines produced. On the whole, however, the most hopeful possibility is (5), viz. that the 2-quantum orbit of the hydrogen atom is metastable. If it spends a large fraction of its life in this state, then the radiation-pressure is increased by $\frac{3}{2}\times 10^3$ at all values of the gas-pressures greater than 10^{-10} atmosphere; and if we take

$$k_\nu = 5\times 10^8$$

and assume that there are six lines effective, the discrepancy is completely removed.

But, whatever be the true reason for this disagreement between theory and observation, there can be no doubt that hydrogen actually is supported at a high level by radiation-pressure; and it therefore

* It is quite certain that the 2_1 orbit actually is metastable; the first Lyman line being due to transitions from the 2_2 orbit only (*vide* Russell and Compton, *Nature*, **114**, 16 (1924). The explanation of the high levels attained by the Balmer lines is then identical with that given by Milne (*M.N.*, **84**, 354) for "orthohelium." The difficulty is that we do not know the relative numbers of the atoms in the 2_1 and 2_2 orbits respectively; if the former number is very small, the explanation breaks down, but this appears to be the only possible reason why it should do so.

seems very probable that the reduced intensity above a sunspot will cause it to flow inwards. The factors by which the intensities are cut down at the various relevant wave-lengths are given in the table.

λ.	6564 (H_α)	4861 (H_β).	3646 (Balmer $_\infty t$).	1215 (1st Lyman).	912 (Lyman $_\infty t$).
$\frac{E_\nu(4500°)}{E_\nu(6000°)}$	$\frac{1}{3}$	$\frac{1}{6{\cdot}4}$	$\frac{1}{12}$	$\frac{3}{5}\times 10^{-3}$	$\frac{1}{2}\times 10^{-4}$

Thus if the main support is derived from the Lyman series, the available energy is reduced almost to zero by the fall from 6000° to 4500° ; whilst if, as is more likely, the radiation-pressure is acting upon atoms in a metastable 2-quantum orbit, it will still be reduced by a factor which is somewhere between $\frac{1}{3}$ and $\frac{1}{12}$.

This seems quite adequate to account for the observed inward flow of the gas.

There is one difficulty in this connection, however, which cannot be very satisfactorily cleared up at present, and that is, that if the velocities of inflow are of the same order as those of thermal agitation, we should expect to find a value for hydrogen some six times as great as for calcium. The actual velocities found are 3·8 km./sec. for Ca+ at 14,000 km., 3·0 km./sec. for H_α at 12,000 km., 2·1 km./sec. for H_β at 8000 km., and so on. But at such pressures as we are dealing with here, collisions are so infrequent that the statistical conception of temperatures is very little help ; we know nothing of the " equivalent " temperatures which correspond to the actual mean velocities of the atoms ; and these may well be different for different elements. The agreement for calcium as regards order of magnitude is therefore possibly fortuitous.

The University, Leeds.

Stellar Parallaxes Determined at the Royal Observatory, Greenwich.

(Communicated by the Astronomer Royal.)

The following list is a continuation of the results obtained with the 26-inch telescope, and published annually in the November number of the *Monthly Notices* since 1920.

In the column Proper Motion the first value given is from meridian observations ; the second is the relative proper motion derived from the photographs. The motions in both R.A. and Dec. are expressed in arc of a great circle.

In calculating the parallax, the relative proper motion has been used, *not* the meridian proper motion.

The Probable Error has been calculated by the formula

$$r = 0{\cdot}6745\times\sqrt{\frac{[vv]}{m-\mu}}\times\frac{1}{\sqrt{w}}.$$

The first P.E. given is that of the parallax, the second is the P.E. of the relative proper motion, and applies to both R.A. and Dec. when the latter is given.

Generally six comparison stars of mean magnitude 11–12 have been used, the magnitude of the parallax star being reduced to equality by means of the rotating sector. The parallax as well as the photographic proper motion is relative to these stars.

No.	Name and B.D. No.	Mag. Vis. Photog.	Spectral Type.	R.A. Dec. 1900·0.	Proper Motion. R.A.	Proper Motion. Dec.	Parallax.	P.E.	No. of Plates.
		m		h m s	″	″	″	″	
1	L. 2933	8·3	G5	1 32 40	−·023	+·242	+·027	±·008	26
	64° 212	8·9		+64° 39′	−·054			±·002	
2	L. 2966	7·4	G5	1 34 11	+·756	−·237	+·044	±·006	25
	66° 145	8·0		+66° 25′	+·672			±·002	
3	56° 731	9·5	Oa	2 44 49			−·001	±·004	35
		10·3		+56° 32′	−·002			±·002	
4	74° 157	9·5		3 24 46	+·168	−·063	−·021	±·005	24
		10·8		+74° 55′	+·132			±·005	
5	O.A. 7267	9·5		6 44 32	+·134	−·201	+·015	±·006	25
	64° 606	9·5		+64° 6′	+·058			±·006	
6	L.F. 1279	8·5	Ko	8 7 54	−·119	−·005	+·027	±·004	25
	65° 622	8·9		+64° 53′	−·059			±·004	
7	67° 591	9·1		9 22 11	−·136	−·090	+·014	±·006	26
		9·8		+67° 2′	−·140			±·003	
8	A.C. 65° 3825	..		12 6 24	−·235	−·220	+·006	±·007	23
		11·9		+65° 36′	−·201			±·005	
9	Gr. 1893	6·5	Ko	12 22 4	−·158	−·011	+·001	±·011	21
	72° 565	7·2		+72° 29′	−·130			±·008	
10	Gr. 2060	6·0	Ko	13 48 32	−·181	−·063	+·021	±·008	28
	69° 724	7·4		+68° 49′	−·195			±·005	
11	L.F. 2404	8·5	Ko	14 1 0	−·123	−·011	+·010	±·007	28
	73° 615	9·2		+73° 25′	−·104			±·004	
12	L.F. 2415	7·5	Ko	14 4 40	+·071	+·121	−·008	±·006	24
	75° 527	8·3		+75° 12′	+·085			±·005	
13	Gr. 2630	8·1	G5	18 34 29	−·035	−·251	·000	±·007	32
	63° 1439	8·3		+63° 37′	−·039			±·005	

No.	Name and B.D. No.	Mag. Vis. Photog.	Spectral Type.	R.A. Dec. 1900·0.	Proper Motion. R.A.	Proper Motion. Dec.	Parallax.	P.E.	No. of Plates.
		m		h m s	″	″	″	″	
14	L. 35182	7·7	Ko	18 44 6	+·053	+·111	+·020	±·010	26
	64° 1289	8·0		+64° 42′	+·039			±·006	
15	O.A. 19154	7·5	Ko	19 11 23	−·030	−·104	+·013	±·010	23
	79° 614	8·5		+79° 29′	−·053			±·008	
16	Rad. 4934	9·0		20 36 11	+·102	+·212	−·001	±·009	25
	80° 662	8·8		+80° 47′	+·068			±·008	
17	O.A. 21338	8·5	K2	20 51 18	·000	−·770	+·130	±·008	25
	61° 2068	9·9		+61° 48′	−·031			±·007	
18	O.A. 21786	8·0	Ko	21 5 52	+·129	−·049	+·011	±·007	27
	69° 1148	8·7		+69° 16′	+·104			±·007	
19	L.F. 3765	8·0	G5	21 16 44	+·200	+·032	−·003	±·008	26
	65° 1573	8·7		+65° 45′	+·134			±·006	
20	O.A. 22853	9·4		21 42 25	+·097	+·221	−·011	±·007	25
	73° 943	10·1		+73° 59′	+·102			±·005	
21	C. 3405	9·5		22 5 19	−·020	−·006	+·001	±·008	23
	85° 379	11·4		+85° 37′	−·011			±·006	
22	75° 839	9·2		22 34 9	−·164	+·152	−·009	±·008	25
		9·2		+75° 58′	−·203			±·005	
23	75° 855	9·0		22 49 54	+·322	+·093	+·028	±·007	25
		9·6		+75° 30′	+·282			±·005	
24	L.F. 4371	7·5	Go	23 1 10	+·590	+·152	+·023	±·009	22
	67° 1498	7·9		+67° 52′	+·591			±·008	
25	O.A. 25629	9·3		23 23 41	−·198	−·039	−·006	±·007	25
	75° 879	10·1		+76° 10′	−·170			±·006	
26	Br. 3181	6·9	G5	23 47 9	+·264	−·087	+·037	±·008	27
	76° 934	6·7		+77° 2′	+·257			±·007	

No. 13. Σ2365.
No. 21. P.M. in *Ast. Cat.*, vol. iv., in Dec. *for* −″·233 *read* −″·006.

The Stellar Magnitude Scales of the Astrographic Catalogue. Twentieth Note. The Sydney Magnitudes for −52° and the Melbourne Magnitudes for −65° and −67°. By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

[*Introductory.*—Since the Nineteenth Note of this series appeared (*M.N.*, **85**, 471), followed by a Provisional General Survey in view of the forthcoming meeting of the Int. Ast. Union (*M.N.*, **85**, 610), a new importance has been given to these counts by the results obtained at Greenwich for effective wave-lengths of stars of different spectral classes (see *M.N.*, **86**, 28, and *Observatory* for 1926 January, pp. 3 and 4). It appears that when ordinary plates are used the effective wave-length changes but little with the spectral type, except for very blue and very red stars. This result is disappointing for those who wish to determine spectral type from effective wave-length : but reassuring to those who have been treating all spectral classes together, as in the case of the Astrographic Catalogue. It might well have been that such mass production would turn out to be waste labour, if the behaviour of the different types had differed appreciably : the Greenwich results are a welcome and unexpected *ex post facto* justification of what (it must be admitted) was a rather rash procedure.]

The Sydney Magnitudes for −52°.

1. One complete zone (−52°) of the Sydney Astrographic Catalogue has been printed in four volumes, dated 1923 (vols. iii. and iv.) and 1925 (vols. i. and ii.). Disturbing news concerning the future of the observatory suggests that further volumes may not be very soon forthcoming, and in any case it seems desirable to examine this first-fruits. The examination is attended with special difficulties because of the meagreness of the information vouchsafed. Were four volumes of astronomical observations ever before published without any indication of the editor's name in any of them ? The plates in the zone are in two series :

The *old series*, taken about the date 1892 under the directorship of H. C. Russell, and sent by him to Melbourne for measurement. Four different instruments were used in the measures : Repsold I., Repsold II., machine " slide," and machine " comb."

The *new series*, taken some twenty-eight years later, about date 1920, under the directorship of W. E. Cooke, and measured at Sydney on machines Sydney A or Sydney B. It will be seen below that this *new series* contains much fewer stars than the *old series*, in spite of certain rather elaborate precautions, such as working only on " perfectly fine " nights, of which details are given in the introduction to volume iii. But the exposures were unfortunately reduced to 4^m, 2^m, and 13^s, and this probably accounts for the loss of stars.

Before attempting to convert the measures on these six different machines (and on several different plans) into a single system, it is desirable to glance at each set separately.

2. *Repsold I.*—Of the 120 plates in the zone, 59 were measured on "Repsold machine I.," and it seems best to take this series as the standard to which the others may be reduced. A striking feature of the measures is the great number entered under the single heading 2·2, representing the faintest stars on the plate. Thus on plate with centre $11^h\ 12^m$ the numbers are

Heading	2·0	2·2	2·3	2·4	2·5	2·6	2·7	2·8	2·9	3·0
Number	1	522	0	12	83	9	0	68	8	48

It seems doubtful whether much importance can be attached to the minor divisions of the scale. If we divide it at 2·3, 3·3, 4·3, we avoid discussion of any "decimal equation," and get groups comparable with that under the single heading 2·2, the members of which cannot, in any case, be distinguished from the recorded measures. (Moreover, these particular headings 2·3, 3·3, 4·3 are practically vacant.) Other breaks were made at 6·1, 7·9, 10·1, 11·9; *i.e.* the groups α, β, γ, δ below included all values *larger* than 11·8, 10·0, 7·8, 6·0. This subdivision was chosen after preliminary trial of a few plates, but is not altogether satisfactory since the few chosen were not quite representative. But the figures for the bright stars are very erratic, and it seems scarcely worth while to do the work all over again.

TABLE I.

Value of log N for Repsold I.

Class.	Limit.	1^h.	2^h.	3^h.	4^h.	7^h.	8^h.	9^h.	10^h.	11^h.	12^h.	13^h.
α	11·8	9·28	9·40	9·40	9·10	0·00	9·70	9·82	8·92	9·77	..	9·77
β	10·0	9·65	9·57	9·57	9·27	0·18	9·88	0·12	8·92	0·03	9·88	9·82
γ	7·8	9·98	9·88	0·18	0·12	0·35	0·48	0·66	0·03	0·59	0·33	0·20
δ	6·0	0·26	0·14	0·33	0·32	0·40	0·69	0·90	0·44	0·92	0·64	0·48
ε	4·3	0·59	0·48	0·75	0·78	0·82	0·97	1·34	1·12	1·25	1·07	0·98
ζ	3·3	0·98	0·94	1·06	1·17	1·21	1·33	1·71	1·64	1·65	1·58	1·48
η	2·3	1·27	1·16	1·39	1·48	1·82	1·82	2·13	2·16	2·02	2·16	2·02
θ	0·0	1·61	1·49	1·68	1·81	2·04	2·10	2·49	2·49	2·33	2·33	2·16
No. of plates		4	2	2	4	1	2	3	3	3	2	3

TABLE I.—*continued.*

Class.	Limit.	14^h.	15^h.	16^h.	17^h.	18^h.	19^h.	20^h.	21^h.	22^h.	23^h.
α	11·8	9·40	0·00	9·50	..	9·74	9·58	9·70	9·95	9·40	9·62
β	10·0	9·40	0·22	9·50	8·92	0·00	9·70	9·77	0·05	9·70	9·77
γ	7·8	0·12	0·65	9·91	9·70	0·44	0·16	0·26	0·40	0·00	0·34
δ	6·0	0·41	0·88	0·20	9·82	0·68	0·50	0·45	0·62	0·18	0·61
ε	4·3	0·94	1·35	0·87	0·56	0·99	0·86	0·72	0·86	0·68	0·98
ζ	3·3	1·43	1·81	1·39	1·08	1·32	1·15	0·99	1·09	0·92	1·28
η	2·3	2·00	2·23	2·04	1·72	1·82	1·65	1·50	1·50	1·49	1·74
θ	0·0	2·17	2·62	2·34	2·05	2·15	1·91	1·75	1·77	1·71	1·93
No. of plates		3	3	4	3	5	4	3	2	1	3

3. The numbers for such plates as are available in each hour were then added together, and the cumulative totals N′ formed. From log N′ was subtracted log (4×number of plates), assuming 4 square degrees per plate, to get log N for each square degree. The values of log N are given on p. 65.

4. It does not seem worth while to give any elaborate correction of so imperfect a series for galactic position: but the main effect of the galaxy can be shewn by the first harmonic, and 12 points read off from the graphs of the above figures give

		Int.
for β, log N =	$9{\cdot}13+{\cdot}03\sin\alpha-{\cdot}02\cos\alpha$	1·11
γ ,,	$0{\cdot}24+{\cdot}05\sin\alpha-{\cdot}18\cos\alpha$	0·24
δ ,,	$0{\cdot}48+{\cdot}03\sin\alpha-{\cdot}18\cos\alpha$	0·41
ϵ ,,	$0{\cdot}89-{\cdot}01\sin\alpha-{\cdot}18\cos\alpha$	0·39
ζ ,,	$1{\cdot}28-{\cdot}03\sin\alpha-{\cdot}32\cos\alpha$	0·47
η ,,	$1{\cdot}75-{\cdot}09\sin\alpha-{\cdot}39\cos\alpha$	0·28
θ ,,	$2{\cdot}03-{\cdot}09\sin\alpha-{\cdot}42\cos\alpha$	

For the first group (α) the blanks at 12^h and 17^h make it difficult to assign values. The intervals between the mean values, shown in the last column, are none of them far from 0·36, except the first.

5. *Repsold II.*—Nine plates, measured as in Repsold I., so that the classes ought to correspond. The values for log N were compared with those interpolated from above table, with the following results :—

TABLE II.

Excess of log N for Repsold II. over Repsold I.

(The unit is 0·01 in log N.)

R.A. h m	α	β	γ	δ	ϵ	ζ	η	θ
5 12	..	..	+13	+15	+ 1	0	− 8	−14
7 36	−20	−40	− 3	+ 7	+26	+16	0	+ 7
8 0	..	−32	−30	−19	+ 2	−11	−25	−39
8 24	+63	+48	+ 4	+19	+38	+20	− 1	− 9
8 36	+53	+38	+ 1	−15	−9	−16	−13	−21
10 48	..	..	−45	−48	−16	−24	− 3	− 5
12 36	..	..	− 2	+ 6	+31	+ 1	− 7	− 3
21 24	−21	−32	+ 3	+10	+11	+12	+ 5	+ 6
22 12	..	..	−34	−18	−18	−37	−54	−47
Mean	+19	− 4	−10	− 5	+ 7	− 4	−12	−14

For the groups γ to θ there is a small negative mean value of −0·06, largely due to the obviously defective plates $8^h\ 0^m$ and $22^h\ 12^m$. We might apply this as a correction, and accordingly increase the numbers of Repsold II., but on the whole it seems better to treat the defect as accidental, and to associate Repsold II. directly with Repsold I.

6. *Machine "Slide."*—Eighteen plates, measured in a similar way to the Repsolds. Dividing the measures, therefore, in a similar way at 2·3, 3·3, 4·3, etc., each of 10 plates was separately compared with the Repsold graph, and the excess Slide – Repsold formed as below. (The other 8 plates were deficient in bright stars.)

TABLE III.

Excess of log N (Slide) over log N (Repsold) in Units of 0·01.

R.A. h m	α.	β.	γ.	δ.	ε.	ζ.	η.	θ.
5 48	+ 7	− 9	−12	− 5	−20	+13	+18	+24
9 0	−14	0	+ 7	+23	+26	+16	+16	+ 9
6 36	−53	−17	+30	+50	+46	+32	+17	+14
7 24	−54	−43	− 1	+10	+18	+10	+43	+16
9 12	−10	−33	−28	−15	− 6	0	+13	+18
9 36	+35	+53	+33	+41	+28	+24	+24	+ 9
11 24	..	−60	−49	−28	−13	+ 3	+31	+24
12 24	..	+27	−30	− 8	− 3	−39	−42	−28
15 12	+26	+37	+24	+59	+44	+22	+10	− 7
21 0	−12	−22	− 2	+ 5	0	+ 2	− 4	−1
Mean	−12	− 7	− 3	+13	+12	+ 8	+13	+ 7

7. Except for the bright stars, which are apt to be erratic, the scale seems to be satisfactorily the same as for the Repsold. There is a small difference of zero point, the slide numbers being systematically greater in the ratio 1·3 to 1 (log 1·30 = 0·11). Hence from the slide numbers one-quarter has been subtracted for association with the Repsolds.

8. After this work had been done, however (when the writer was away from references), the paper on the Melbourne magnitudes (*M.N.*, 77, 35) was consulted, and it was seen how curiously different were the results then obtained for the "slide" machine. The excess of log N for slide over the mean of the two Repsold results is *greatest* for the bright stars, running down from +0·48 to +0·04 for "All," much as seen in the case of plate $9^h\ 36^m$ in the above table (p. 37, *loc. cit.*). Enquiries were made of Melbourne whether any light could be thrown on the discrepancy, and Mr. Baldwin kindly had three plates which had been previously measured on S remeasured on R_2. But this experiment gave no help, for the difference disappeared. Possibly it was due to some particular time or measurer; but since no details of this kind are supplied in the Sydney volumes, it is impossible to investigate the matter from them.

9. *Machine "Comb."*—Seven plates; diameters measured in figures for which a preliminary trial suggested the following as nearly corresponding to the Repsold divisions (omitting α): 18, 14, 10, 7, 5·5, 4·0, 0·0.

The separate plates gave the following results :—

TABLE IV.

Log N (Comb)—log N (Repsold).

R.A. h m	β.	γ.	δ.	ϵ.	ζ.	η.	θ
2 0	+10	+27	+15	+ 1	0	− 2	−12
2 48	..	..	0	+26	+23	+24	+ 1
4 12	−27	−74	−31	−29	−30	+ 4	+ 8
5 36	..	..	− 5	+11	+14	+ 8	+ 9
7 48	−66	−52	− 2	+17	+ 4	−12	+ 5
12 0	−30	−59	−31	−36	−62	−54	− 8
17 24	..	..	+70	+55	+24	−23	−52
	−28	−40	+ 2	+ 6	− 4	− 8	− 7

The means suggest that, except perhaps for the brightest two groups, the equivalence is sufficiently good.

We now come to the *new series* of plates, taken a quarter of a century after the *old series*.

10. *Sydney Machine A* (21 plates).—The images are compared with type images lettered *a* to *l* and assigned either definitely to one of these types, or to an intermediate type *ab*, *bc*, *cd*, etc. Besides, there are a (very) few stars brighter than *a*. In all there are 22 classes, for which separate counts were made.

Preliminary investigation shewed that the alphabetic scale is a nearly linear scale in log N, with the important exception that the intermediate types *ab*, *bc*, etc., do not fall evenly between the simple types *a* and *b*, etc. It will suffice to give the counts for 4 groups of plates to shew the consistency of this character, as in Table V. The groups A, B, C, D are as follows :—

A. h m	B. h m	C. h m	D. h m
3 12	13 12	16 36	22 0
5 0	13 24	17 0	22 24
5 24	14 0	19 12	22 48
6 12	14 48	20 24	23 12
6 24	15 48	20 36	23 48
10 0			

In forming log N, we subtract from the log N1 for the first group log $6\times4 = 1{\cdot}38$, and from the other three groups log $5\times4 = 1{\cdot}30$.

TABLE V.

	Simple Counts.				log N (Cumulative).				Differences.			
	A.	B.	C.	D.	A.	B.	C.	D.	A.	B.	C.	D.
]*a*	9	7	3	2	9·57	9·55	9·18	9·00				
a	3	20	3	3	9·70	0·13	9·48	9·40	13	58	30	40
ab	1	4	2	1	9·73	0·19	9·60	9·48	3	6	12	8
b	9	14	7	6	9·96	0·35	9·88	9·78	23	16	28	30
bc	5	7	3	2	0·05	0·42	9·96	9·85	9	7	8	7
c	14	40	6	6	0·22	0·66	0·08	0·00	17	24	12	15
cd	8	13	7	4	0·31	0·72	0·19	0·08	9	6	11	8
d	18	53	26	10	0·45	0·90	0·46	0·23	14	18	27	15
de	6	23	9	7	0·48	0·96	0·52	0·31	3	6	6	8
e	27	94	31	20	0·62	1·14	0·69	0·49	14	18	17	18
ef	32	61	41	12	0·74	1·23	0·84	0·56	12	9	15	7
f	56	150	80	34	0·89	1·39	1·04	0·73	15	16	20	17
fg	28	62	42	16	0·95	1·44	1·12	0·79	6	5	8	6
g	91	276	97	49	1·11	1·62	1·25	0·94	16	18	13	15
gh	49	129	116	21	1·17	1·68	1·37	0·99	6	6	12	5
h	131	362	160	53	1·31	1·82	1·50	1·09	14	14	13	10
hj	44	114	98	22	1·35	1·86	1·56	1·13	4	4	6	4
j	185	461	277	102	1·47	1·98	1·70	1·27	12	12	14	14
jk	104	129	167	55	1·53	2·01	1·77	1·33	6	3	7	6
k	225	281	361	93	1·64	2·06	1·89	1·41	11	5	12	8
kl	23	11	36	11	1·65	2·06	1·90	1·42	1	0	1	1
l	4	7	31	12	1·65	2·07	1·90	1·43	0	1	0	1

The differences in the last columns are between consecutive values of log N, in units of 0·01.

11. Omitting from consideration the top three and bottom two we see that the differences in the last columns are alternately large and small. But there seems to be a second alternation impressed on this : let us take the sums in fours, of which the first member heads the column.

TABLE VI.

	bc−*b*.	*c*−*bc*.	*cd*−*c*.	*d*−*cd*.
	31	68	34	74
	23	67	43	68
	25	62	29	51
	18	52	22	36
Sum	97	249	128	229
Mean	6	16	8	14

12. The average of the intervals *b* to *c*, *d* to *e*, *f* to *g*, and *h* to *j* is thus the same as that of the intervals *c* to *d*, *e* to *f*, *g* to *h*, and *j* to *k*, viz. 22 units of 100 log N : but the intermediate steps divide the former set after 6 units (instead of 11) and the latter at 8 units. To get a uniform scale we should thus add 5 units in the former case and 3 in the latter. There are sensible deviations from these means in the four separate sets, but it seems needless to follow them further at present.

13. The important point for us is the conversion of this scale into that of Repsold I., and attention is arrested at the outset by the smallness of the totals (class *l*) as compared with those of the *old series*. With the improvement in sensitiveness of plates during the twenty-eight years 1892–1920, and the special precautions taken to retain for measurement only the best plates taken, one would have expected log N to be greater. But its mean value for Repsold I. is 2·03, whereas for the groups A, B, C, D the mean value is only 1·76. In other words, the average number of stars measured with the Repsold machine is 428, with Sydney A only 230. (Even the reduction of the exposure times from 6^m, 3^m, 20^s to 4^m, 2^m, 13^s scarcely accounts for the difference.) This rough method, however, slightly overestimates the systematic difference. Forming approximate values of the Repsold scale for the four groups, we find that it compares with the Sydney A scale as follows :—

TABLE VII.

Comparison of "Sydney A" Scale with Repsold.

(Unit 0·01 in log N.)

Repsold.	β.	γ.	δ.	ϵ.	ζ.	η.	θ.
A	$a-4$	$b+28$	$c+19$	$e+22$	$g+14$	$j+22$	$l+33$
B	$a-39$	$b-10$	$c-12$	$e-9$	$g-8$	$j+10$	$l+23$
C	$a+18$	$b+20$	$c+26$	$e+9$	$g-10$	$j+1$	$l+10$
D	$a+20$	$b+27$	$c+30$	$e+26$	$g+11$	$j+28$	$l+37$
Mean	$a-1$	$b+16$	$c+16$	$e+12$	$g+2$	$j+15$	$l+26$
Alternative	$]a+17$	$ab+40$	$bc+33$	$de+29$	$fg+18$	$hj+28$	

14. In the last line is shewn a possible alternative for the conversions, if we use the intermediate groups *ab*, *bc*, etc. If then we adopt for use

$$]a+17,\ b+16,\ c+33,\ de+29,\ fg+18,\ hj+28,\ l+26,$$

then the mean of the additions is +24, corresponding to log 1·75. It is therefore concluded that the numbers for the groups $]a$, *b*, *c*, *de*, *fg*, *hj*, *l* multiplied by 1·75 may be considered equivalent to the Repsold groups β, γ, δ, etc.

15. *Sydney Machine B.*—Five plates only. After the bright stars, treated specially, classes 1, 2, etc., to 12 are given, with intermediates 1–2, 2–3, etc., much as for the classes *a*, *b* . . . *l*, and intermediates

with machine A. The scale is again found to be fairly uniform in log N, but with an interval of approximately 0·12 between classes, much smaller than the 0·22 for machine A. The separate counts are so few that they may be given in full, omitting the intermediate intervals. The bright stars are divided into two classes: P with numbers over 19, and Q with 19 and under.

TABLE VIII.

Comparison of "Sydney B" Scale with Repsold.

Class.	h m 0 24.	h m 1 24.	h m 2 36.	h m 6 48.	h m 7 0.	Sum.	N′.	log N′ −1·30.
P	1	0	2	2	3	8	8	9·60
Q	1	0	2	4	4	11	19	9·98
1	2	4	1	13	11	31	50	0·40
2	2	4	1	13	18	38	88	0·64
3	4	3	5	13	21	46	134	0·83
4	8	9	5	24	24	70	204	1·01
5	7	4	10	21	37	79	283	1·15
6	9	8	1	21	38	77	360	1·26
7	10	8	12	24	38	92	452	1·36
8	9	14	10	61	43	137	589	1·47
9	9	10	5	31	83	138	727	1·56
10	21	20	30	56	62	189	916	1·66
11	9	9	7	41	1	67	983	1·69
12	0	1	0	1	0	2	985	1·69

16. Comparing with Repsold we find the approximate values

$$\beta = \text{P}+\cdot 20,\ \gamma = \text{Q}+\cdot 15,\ \delta = (1)-\cdot 03,\ \epsilon = (3)-\cdot 03,\ \zeta = (5)+\cdot 03,$$
$$\eta = (8)+\cdot 03,\ \theta = (12)+\cdot 09.$$

These conversions, omitting the numerical additions, were used in the five cases.

17. Summarising, we have the following relations between the results of the six machines. It seems advisable to drop the brightest group α, originally introduced for Repsold I., as it is in several hourly groups unrepresented in the case of other machines :—

Repsold II. ; as for Repsold I., at 10·0, 7·8, 6·0, 4·3, 3·3, 2·3, all.
"Slide" ; as for Repsold I., at 10.0, 7·8, 6·0, 4·3, 3·3, 2·3, all. *Less one-quarter in each case.*
"Comb" ; at 18, 14, 10, 7, 5·5, 4·0, all.
Sydney A ; at]a, b, c, de, fg, hj, all ; *multiplied by 1·75* in each case.
Sydney B ; at P, Q, (1), (3), (5), (8), all.

In each case the seventh group, θ, consists of all the stars on the plate.

18. Using these correspondences, all the counts were converted into terms of Repsold I., and the totals for each hour are given in Table IX.

TABLE IX.

Sydney Zone −52°.

Total Stars Estimated in Each Hour of R.A. (5 Plates).

h	β.	γ.	δ.	ε.	ζ.	η.	θ.	Gal. Lat.
0	8	14	35	67	137	321	495	−64·5
1	8	23	35	72	155	350	740	−63·0
2	7	16	33	79	193	338	591	−57·5
3	4	18	26	65	185	441	795	−50·0
4	4	22	37	108	262	606	1316	−41·5
5	7	30	49	109	299	741	1363	−31·0
6	12	30	74	173	368	924	1663	−22·5
7	14	35	68	205	432	1238	2409	−14·0
8	26	56	92	227	426	1129	1936	− 6·5
9	31	89	170	418	1014	2737	5782	0·0
10	6	27	67	235	739	2536	5254	+ 5·5
11	14	51	112	258	718	2034	3994	+ 9·0
12	9	32	76	239	581	1900	3508	+10·5
13	10	43	81	223	725	2140	3291	+10·0
14	12	34	85	200	652	2014	3196	+ 7·0
15	30	107	216	524	1309	3439	6959	+ 2·5
16	5	18	37	152	520	2146	4441	− 3·5
17	1	11	40	133	360	1089	2267	−11·0
18	20	55	95	193	413	1332	2833	−19·5
19	11	30	57	134	285	875	1662	−28·5
20	9	31	45	89	255	758	1383	−37·5
21	13	37	67	116	219	574	1088	−46·5
22	6	26	40	81	172	436	800	−55·0
23	7	28	56	139	322	846	1461	−61·5

19. There are 5 plates in each hour, or about 20 square degrees. Thus log 20 = 1·30 was subtracted from each log total of Table IX., and the results were then cleared for galactic latitude effect according to the method given in *M.N.*, **78**, 57, and uniformly adopted in subsequent papers. All the seven groups were then analysed harmonically and the results are as in Table X.

20. To correspond with other zones we deduce the rate of change of these harmonics for 1·00 in log N_0 by subtracting the sum of the rows for γ and ϵ from the sum for η and All, as shown at the foot of the table. Attention has been specially called to the first harmonic, the difference for which is

$$-\cdot053 \sin\alpha - \cdot067 \cos\alpha = -\cdot086 \cos(\alpha - 38^\circ).$$

The negative coefficient is used in order to represent an obscuration

effect; and for the maximum (38°) the formula was suggested (for a spiral)

$$\alpha = 247^\circ - 3{\cdot}66\delta = 438^\circ \text{ for } \delta = -52^\circ.$$

The accordance between (438° or) 78° and 38° is fair, but several zones have proved discordant and it seems doubtful whether the hypothesis will ultimately stand.

TABLE X.

Harmonic Analysis for −52°.

Class.	log N_0.	sinα.	cosα.	sin2α.	cos2α.	sin3α.	cos3α.	sin4α.	cos4α.
β	9·84	+·061	+·057	−·111	+·079	−·073	+·069	+·068	−·076
γ	0·40	+·020	+·073	−·068	+·035	−·061	−·008	−·018	−·047
δ	0·72	−·003	+·047	−·058	+·057	−·032	+·015	−·017	+·015
ε	1·14	+·010	+·015	−·033	+·043	−·008	+·031	−·032	+·007
ζ	1·56	−·004	−·020	+·025	+·048	−·008	+·003	−·044	−·002
η	2·02	−·058	−·055	+·002	+·042	−·015	−·001	−·055	+·002
All	2·31	−·059	−·044	+·011	+·022	+·013	−·010	−·075	−·002
Diff.	2·79	−·147	−·187	+·114	−·014	+·067	−·034	−·080	+·040
	1·00	−·053	−·067	+·041	−·005	+·024	−·012	−·029	+·014

21. In *M.N.*, **85**, 610, a preliminary discussion is given, not of the rates of change of the harmonics, but of their mean values. The diagrams on p. 622 indicate for −52° the following mean values (C) those found in Table X. being given as (O).

Comparison of Prediction with Observation.

	sinα.	cosα.	sin2α.	cos2α.	sin3α.	cos3α.	sin4α.	cos4α.
(C)	+·07	−·05	−·02	−·05	+·03	+·03	+·03	+·02
(O)	−·02	·00	−·02	+·04	−·02	·00	−·05	−·01

There is practically no agreement at all, there being five cases of difference in sign as against only three of similarity. Especially noticeable is the large negative sign for the coefficient of sin4α, which was positive with remarkable persistence in forty other zones (p. 612, *loc. cit.*). It is therefore of special interest to examine two Melbourne zones of which counts have been supplied.

The Melbourne Magnitudes for −65° and −67°.

22. Cumulative totals for each hour of −65° were kindly supplied by Mr. Baracchi in 1916, and are printed on p. 39 of *M.N.*, **77**. But at that time the process since (*M.N.*, **78**, 57) adopted for clearing out the condensation in galactic latitude had not been formulated. This procedure has now been applied and the results are as follows, together with similar results for −67° zone supplied subsequently, the counts for which are given in Table XIII. below.

TABLE XI.

Harmonic Analysis for Melbourne Zone −65°.

d.	log N_0.	sinα.	cosα.	sin2α.	cos2α.	sin3α.	cos3α.	sin4α.	cos4α.
35	0·25	+·045	−·094	·000	−·029	+·033	+·066	+·003	−·091
25	0·68	+·043	−·106	−·009	+·016	−·029	−·034	−·048	−·047
15	1·37	+·024	−·063	+·050	−·059	−·039	+·027	−·056	−·053
10	1·91	+·016	−·037	+·042	−·048	−·026	−·009	−·050	−·039
All	2·23	−·059	−·114	−·022	−·061	−·004	+·020	−·064	−·021
	2·09	−·110	+·018	−·021	−·066	+·038	+·018	−·010	+·040
	1·00	−·053	+·009	−·010	−·032	+·018	+·009	−·005	+·019

TABLE XII.

Harmonic Analysis for Melbourne Zone −67°.

d.	log N_0.	sinα.	cosα.	sin2α.	cos2α.	sin3α.	cos3α.	sin4α.	cos4α.
35	0·19	+·008	−·180	−·105	+·008	−·071	−·052	+·109	−·014
25	0·67	+·029	−·088	−·089	+·063	−·056	−·029	+·033	−·024
15	1·39	+·033	−·036	−·064	+·017	−·095	−·039	−·059	−·004
10	1·91	+·014	−·052	−·064	+·017	−·091	−·015	−·038	+·015
All	2·27	+·042	−·072	−·014	−·008	−·071	−·007	−·048	+·048
	2·12	−·006	·000	+·075	−·071	−·011	+·046	−·062	+·091
	1·00	−·003	·000	+·035	−·033	−·005	+·022	−·029	+·043

TABLE XIII.

Cumulative Totals (N′) for Melbourne −67° for Each Hour of R.A.

d. >	40.	35.	30.	25.	20.	15.	10.	All.	Gal. Lat.
h									°
0	4	8	11	20	25	92	264	456	−50·0
1	3	8	12	13	21	80	237	485	−49·0
2	2	3	10	19	23	55	143	403	−46·5
3	3	9	15	19	30	83	193	491	−42·5
4	1	3	8	15	30	91	306	592	−37·5
5	6	11	14	23	45	193	702	2014	−32·0
6	4	15	27	42	77	218	539	1251	−26·0
7	19	37	58	90	112	198	648	1171	−20·5
8	5	10	17	24	51	135	441	1126	−15·5
9	8	14	20	41	76	236	672	1570	−11·0
10	19	38	73	118	165	462	1578	2875	− 7·5
11	11	26	43	66	114	316	1006	2504	− 5·5
12	19	31	52	89	144	334	1116	2797	− 4·5
13	17	36	47	71	114	275	997	2160	− 5·0
14	12	21	32	52	86	278	906	1789	− 7·0
15	7	20	39	62	88	272	1025	2376	−10·0

TABLE XIII.—*continued.*

d.> h	40.	35.	30.	25.	20.	15.	10.	All.	Gal. Lat. °
16	3	9	15	27	47	197	552	1475	−14·0
17	3	6	12	20	46	150	353	1069	−18·5
18	3	10	12	19	38	97	393	799	−24·0
19	9	20	29	37	56	131	436	608	−30·0
20	3	14	21	33	49	143	366	686	−35·5
21	1	4	6	16	32	121	333	643	−40·5
22	3	8	14	27	51	136	371	648	−45·0
23	2	4	9	18	27	71	225	434	−48·5

23. As regards the first harmonic of the differences at the ends of Tables XI. and XII. we have

$$\text{for } -65°: -\cdot053 \sin\alpha + \cdot009 \cos\alpha = -\cdot054 \cos(\alpha - 99°),$$
$$\text{for } -67°: -\cdot003 \sin\alpha + \cdot000 \cos\alpha = -\cdot003 \cos(\alpha - 90°),$$

the values for the maximum given by the formula $\alpha = 247° - 3{\cdot}66\delta$ being 125° and 132° respectively. The values of O - C for the three zones −52°, −65°, and −67° are thus 38°−78°, 99°−125°, 90°−132°, or −40°, −26°, and −42°, which are in fair accord. But the coefficients are so small that the accordance cannot be taken very seriously.

24. As regards the mean values (giving the coefficients for log N_0 = 1·50 approx.) we may compare the values (C) shewn by the diagrams on p. 622 of *M.N.*, **85**, for −66° with those given in Tables XI. and XII.

	sinα.	cosα.	sin2α.	cos2α.	sin3α.	cos3α.	sin4α.	cos4α.
(C)	+·12	−·08	−·01	−·03	+·03	+·03	+·01	+·01
−65°	+·01	−·08	+·01	−·03	−·01	+·01	−·04	−·05
−67°	+·02	−·08	−·07	+·02	−·08	−·03	·00	+·01

The accordance with prediction is better than that for the Sydney zone, but still leaves much to be desired.

Summary.

(*a*) The Sydney zone −52°, the only one hitherto published, contains measures of plates taken (at Sydney) about 1892 and measured at Melbourne on four different machines, and others taken some twenty-eight years later and measured at Sydney on two different machines. There is some difficulty in co-ordinating the various estimates of magnitude, but an attempt is made to do so, and to analyse the results harmonically after correcting approximately for galactic condensation.

(*b*) Two Melbourne zones −65° and −67° are also analysed.

(*c*) The harmonic coefficients obtained do not agree well with those indicated by the Provisional General Survey (*M.N.*, **85**, 610). It seems a fair inference that the absence of zones near the South Pole made an essential difference to the analysis.

On the Fundamental Equations of Planetary Motion.

By W. M. Smart, M.A., D.Sc.

In the first part of the paper, §§ 1–7, certain relations amongst the derivatives of the disturbing function with respect to the six elliptic elements are obtained, and in the second part, §§ 8–14, these relations are employed to derive the six well-known equations of planetary motion.

§ 1. The rectangular co-ordinates of the planet under consideration are (x, y, z), and of the disturbing planet (x_1, y_1, z_1). The latter are supposed known and expressed in terms of t and six constants. The components of the velocity of the planet are x', y', z'. The equations * of motion of the planet are, in Tisserand's notation,

$$\frac{dx}{dt}-\frac{\partial \mathrm{H}}{\partial x'}=0 \quad : \quad \frac{dx'}{dt}+\frac{\partial \mathrm{H}}{dx}+\mathrm{X}=0 \quad . \quad . \quad . \quad (1)$$

with two similar sets of equations in y and z.

We write

$$\mathrm{X}=-\frac{\partial \mathrm{R}}{\partial x} \quad : \quad \mathrm{Y}=-\frac{\partial \mathrm{R}}{\partial y} \quad : \quad \mathrm{Z}=-\frac{\partial \mathrm{R}}{\partial z} \quad . \quad . \quad . \quad (2)$$

R being the disturbing function.

If, in equations (1), R is neglected, the solutions are those of simple elliptic motion, and they may be written symbolically as

$$x=x(t\,;\ a_1, a_2, a_3\,;\ b_1, b_2, b_3) \quad . \quad . \quad . \quad (3)$$

with corresponding forms in y and z, and

$$x'=x'(t\,;\ a_1, a_2, a_3\,;\ b_1, b_2, b_3) \quad . \quad . \quad . \quad (4)$$

with corresponding forms in y' and z'.

Here the a's and the b's are the six elliptic elements; they are defined, with reference to the usual notation, as follows :—

$a_1 \equiv a$, semi-major axis.
$a_2 \equiv e$, eccentricity.
$a_3 \equiv \chi$, Kepler's equation being $u-e \sin u = nt+\chi$.
$b_1 \equiv \theta$, longitude of the ascending node.
$b_2 \equiv \omega$, the angle measured in the plane of the orbit from the ascending node to perihelion.
$b_3 \equiv i$, the inclination of the orbital plane with respect to the ecliptic.

Regarding now the a's and b's as variables, we have from (1) the six equations which form the basis of Lagrange's method of the variation of constants, viz.

$$\sum_{s=1}^{3}\frac{\partial x}{\partial a_s}\frac{da_s}{dt}+\sum_{s=1}^{3}\frac{\partial x}{\partial b_s}\frac{db_s}{dt}=0 \quad . \quad . \quad . \quad (5)$$

* *Mécanique Céleste*, Tome i., 173.

with two similar equations in y and z, and

$$\sum_{s=1}^{3}\frac{\partial x'}{\partial a_s}\frac{da_s}{dt}+\sum_{s=1}^{3}\frac{\partial x'}{\partial b_s}\frac{db_s}{dt}+\mathrm{X}=0 \quad . \quad . \quad . \quad (6)$$

with two similar equations in y' and z'.

§ 2. Take axes $O\xi$, $O\eta$ in the plane of the instantaneous elliptic orbit such that $O\xi$ is in the direction of perihelion, and let $O\zeta$ be normal to this plane. Let (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) be the direction-cosines of $O\xi$, $O\eta$, $O\zeta$ respectively with reference to the x, y, z axes. Then we have

$$\left.\begin{aligned}
l_1 &= \cos b_1 \cos b_2 - \sin b_1 \sin b_2 \cos b_3 \\
l_2 &= -\cos b_1 \sin b_2 - \sin b_1 \cos b_2 \cos b_3 \\
l_3 &= \sin b_1 \sin b_3 \\
m_1 &= \sin b_1 \cos b_2 + \cos b_1 \sin b_2 \cos b_3 \\
m_2 &= -\sin b_1 \sin b_2 + \cos b_1 \cos b_2 \cos b_3 \\
m_3 &= -\cos b_1 \sin b_3 \\
n_1 &= \sin b_2 \sin b_3 \quad : \quad n_2 = \cos b_2 \sin b_3 \quad : \quad n_3 = \cos b_3
\end{aligned}\right\} \quad (7)$$

Then if $(\xi, \eta, 0)$ are the co-ordinates of the planet, referred to the ξ, η, ζ axes, we have

$$x = l_1\xi + l_2\eta\,; \qquad y = m_1\xi + m_2\eta\,; \qquad z = n_1\xi + n_2\eta \quad . \quad (8)$$

Here the direction cosines $l_1, \ldots n_2$ are functions of b_1, b_2, b_3, and ξ and η are functions of t, a_1, a_2, a_3.

From equations (7), $\dfrac{\partial l_1}{\partial b_1}$, etc., are easily calculated, and we have, using these, from (8),

$$\left.\begin{aligned}
\frac{\partial x}{\partial b_1} &= -y \quad : \quad \frac{\partial x}{\partial b_2} = l_2\xi - l_1\eta \quad : \quad \frac{\partial x}{\partial b_3} = l_3\mathrm{K} \\
\frac{\partial y}{\partial b_1} &= \;\; x \quad : \quad \frac{\partial y}{\partial b_2} = m_2\xi - m_1\eta \quad : \quad \frac{\partial y}{\partial b_3} = m_3\mathrm{K} \\
\frac{\partial z}{\partial b_1} &= \;\; 0 \quad : \quad \frac{\partial z}{\partial b_2} = n_2\xi - n_1\eta \quad : \quad \frac{\partial z}{\partial b_3} = n_3\mathrm{K}
\end{aligned}\right\} \quad . \quad (9)$$

where

$$\mathrm{K} = \xi \sin b_2 + \eta \cos b_2 \quad . \quad . \quad . \quad . \quad (10)$$

If v is the true anomaly, and u is the eccentric anomaly, we have the group of equations

$$\xi = r\cos v\,; \qquad \eta = r\sin v \quad . \quad . \quad . \quad (11)$$

$$\xi = a(\cos u - e)\,; \qquad \eta = a\sqrt{1-e^2}\sin u \quad . \quad . \quad (12)$$

$$r = a(1-e\cos u)\,; \qquad u - e\sin u = nt + \chi \quad . \quad . \quad (13)$$

From these and (10), we have

$$K = r \sin (v+b_2) \quad . \quad . \quad . \quad . \quad (14)$$

§ 3. When the co-ordinates x_1, y_1, z_1 of the disturbing planet—which appear in the disturbing function R—are expressed in terms of t, R becomes simply a function of t, x, y, z. Substituting, then, the analytic expressions of x, y, z given by (3), R becomes finally a function of t; a_1, a_2, a_3; b_1, b_2, b_3. We have then

$$\frac{\partial R}{\partial b_1} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial b_1} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial b_1} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial b_1}$$

or

$$X\frac{\partial x}{\partial b_1} + Y\frac{\partial y}{\partial b_1} + Z\frac{\partial z}{\partial b_1} + \frac{\partial R}{\partial b_1} = 0$$

Similarly,

$$X\frac{\partial x}{\partial b_2} + Y\frac{\partial y}{\partial b_2} + Z\frac{\partial z}{\partial b_2} + \frac{\partial R}{\partial b_2} = 0 \quad . \quad . \quad . \quad (15)$$

and

$$X\frac{\partial x}{\partial b_3} + Y\frac{\partial y}{\partial b_3} + Z\frac{\partial z}{\partial b_3} + \frac{\partial R}{\partial b_3} = 0$$

Let Δ_b denote

$$\frac{\partial(x, y, z)}{\partial(b_1, b_2, b_3)}.$$

Then, solving for X, Y, Z, we have

$$-\Delta_b X = \frac{\partial(y, z)}{\partial(b_2, b_3)}\frac{\partial R}{\partial b_1} + \frac{\partial(y, z)}{\partial(b_3, b_1)}\frac{\partial R}{\partial b_2} + \frac{\partial(y, z)}{\partial(b_1, b_2)}\frac{\partial R}{\partial b_3} \quad . \quad (16)$$

with two symmetrical equations for Y and Z.
Now

$$\Delta_b = \begin{vmatrix} -y, & l_2\xi - l_1\eta, & l_3 K \\ x, & m_2\xi - m_1\eta, & m_3 K \\ 0, & n_2\xi - n_1\eta, & n_3 K \end{vmatrix}$$

and it is easily seen that

$$\Delta_b = 0,$$

so that equation (16) becomes

$$\frac{\partial(y, z)}{\partial(b_2, b_3)}\frac{\partial R}{\partial b_1} + \frac{\partial(y, z)}{\partial(b_3, b_1)}\frac{\partial R}{\partial b_2} + \frac{\partial(y, z)}{\partial(b_1, b_2)}\frac{\partial R}{\partial b_3} = 0 \quad . \quad . \quad (17)$$

The vanishing of the Jacobian Δ_b implies the existence of a functional relationship, not involving the b's, amongst the variables x, y, z. Actually, we have

$$\begin{aligned} x^2+y^2+z^2 &= r^2 \\ &= a^2(1-e\cos u)^2 \\ &= \text{function of } t, a_1, a_2, a_3. \end{aligned}$$

This is independent of the b's.

We obtain readily the following :—

$$\frac{\partial(y,\,z)}{\partial(b_2,\,b_3)} = \mathrm{K}x\,; \qquad \frac{\partial(y,\,z)}{\partial(b_3,\,b_1)} = -n_3\mathrm{K}x\,; \qquad \frac{\partial(y,\,z)}{\partial(b_1,\,b_2)} = x(n_2\xi - n_1\eta) \quad (18)$$

Also

$$\mathrm{K} = r\sin(v+b_2)$$

and

$$n_2\xi - n_1\eta = r\sin b_3\cos(v+b_2),$$

and consequently equation (17) may be written

$$\frac{\partial \mathrm{R}}{\partial b_1} - \cos b_3\frac{\partial \mathrm{R}}{\partial b_2} + \sin b_3\cot(v+b_2)\frac{\partial \mathrm{R}}{\partial b_3} = 0 \quad . \quad . \quad (19)$$

or, in the conventional notation,

$$\frac{\partial \mathrm{R}}{\partial \theta} - \cos i\frac{\partial \mathrm{R}}{\partial \omega} + \sin i\cot(v+\omega)\frac{\partial \mathrm{R}}{\partial i} = 0 \quad . \quad . \quad (\mathrm{A})$$

This is the first of the relations involving the differential coefficients of the disturbing function with respect to the elliptic elements.

§ 4. Proceeding as in the previous paragraph, but differentiating with respect to the a's, we obtain the group of equations :—

$$\left.\begin{aligned} \mathrm{X}\frac{\partial x}{\partial a_1} + \mathrm{Y}\frac{\partial y}{\partial a_1} + \mathrm{Z}\frac{\partial z}{\partial a_1} + \frac{\partial \mathrm{R}}{\partial a_1} &= 0 \\ \mathrm{X}\frac{\partial x}{\partial a_2} + \mathrm{Y}\frac{\partial y}{\partial a_2} + \mathrm{Z}\frac{dz}{\partial a_2} + \frac{\partial \mathrm{R}}{\partial a_2} &= 0 \\ \mathrm{X}\frac{\partial x}{\partial a_3} + \mathrm{Y}\frac{\partial y}{\partial a_3} + \mathrm{Z}\frac{\partial z}{\partial a_3} + \frac{\partial \mathrm{R}}{\partial a_3} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (20)$$

Let Δ_a denote

$$\frac{\partial(x,\,y,\,z)}{\partial(a_1,\,a_2,\,a_3)}.$$

Then, solving for X, Y, and Z as before, we obtain

$$-\Delta_a\mathrm{X} = \frac{\partial(y,\,z)}{\partial(a_2,\,a_3)}\frac{\partial \mathrm{R}}{\partial a_1} + \frac{\partial(y,\,z)}{\partial(a_3,\,a_1)}\frac{\partial \mathrm{R}}{\partial a_2} + \frac{\partial(y,\,z)}{\partial(a_1,\,a_2)}\frac{\partial \mathrm{R}}{\partial a_3} \quad . \quad (21)$$

with two symmetrical equations in Y and Z.

Since l_1, l_2, l_3, etc., are functions of the b's only, and ξ, η are functions of the a's only (and t), the determinant, Δ_a is easily seen to be the product of

$$\begin{vmatrix} \frac{\partial \xi}{\partial a_1}, & \frac{\partial \xi}{\partial a_2}, & \frac{\partial \xi}{\partial a_3} \\ \frac{\partial \eta}{\partial a_1}, & \frac{\partial \eta}{\partial a_2}, & \frac{\partial \eta}{\partial a_3} \\ 1\,, & 1\,, & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} l_1\,, & l_2\,, & 0 \\ m_1, & m_2, & 0 \\ n_1\,, & n_2\,, & 0 \end{vmatrix}$$

that is,

$$\Delta_a = 0.$$

Again, there is a functional relationship amongst the variables x, y, z which does not explicitly contain the a's. It is

$$l_3x+m_3y+n_3z=0 \quad . \quad . \quad . \quad . \quad (22)$$

which expresses simply the fact that the instantaneous orbit lies in a plane.

Equation (21) becomes

$$\frac{\partial(y,z)}{\partial(a_2,a_3)}\frac{\partial \mathrm{R}}{\partial a_1}+\frac{\partial(y,z)}{\partial(a_3,a_1)}\frac{\partial \mathrm{R}}{\partial a_2}+\frac{\partial(y,z)}{\partial(a_1,a_2)}\frac{\partial \mathrm{R}}{\partial a_3}=0 \quad . \quad (23)$$

Now,

$$\frac{\partial(y,z)}{\partial(a_2,a_3)}=\begin{vmatrix} m_1, & m_2 \\ n_1, & n_2 \end{vmatrix}\frac{\partial(\xi,\eta)}{\partial(a_2,a_3)}$$

$$=l_3\frac{\partial(\xi,\eta)}{\partial(a_2,a_3)} \quad . \quad . \quad . \quad . \quad (24)$$

and we obtain from (23)

$$\frac{\partial(\xi,\eta)}{\partial(a_2,a_3)}\frac{\partial \mathrm{R}}{\partial a_1}+\frac{\partial(\xi,\eta)}{\partial(a_3,a_1)}\frac{\partial \mathrm{R}}{\partial a_2}+\frac{\partial(\xi,\eta)}{\partial(a_1,a_2)}\frac{\partial \mathrm{R}}{\partial a_3}=0 \quad . \quad . \quad (25)$$

Using the equations (12) and (13), we obtain, after some reductions, and reverting to the usual notation,

$$\left.\begin{aligned} \frac{\partial(\xi,\eta)}{\partial(a_2,a_3)} &= -a^2(\cos u+e)(1-e^2)^{-\frac{1}{2}} \\ \frac{\partial(\xi,\eta)}{\partial(a_3,a_1)} &= -a(1-e^2)^{\frac{1}{2}} \\ \frac{\partial(\xi,\eta)}{\partial(a_1,a_2)} &= (1-e^2)^{-\frac{1}{2}}\left[a\sin u(2-e\cos u-e^2)+ar\frac{\partial u}{\partial a}(\cos u+e)\right] \end{aligned}\right\} \quad (26)$$

We thus have, from (25),

$$a(\cos u+e)\left\{r\frac{\partial u}{\partial a}\frac{\partial \mathrm{R}}{\partial a_3}-a\frac{\partial \mathrm{R}}{\partial a_1}\right\}-(1-e^2)\frac{\partial \mathrm{R}}{\partial a_2}+\sin u(2-e\cos u-e^2)\frac{\partial \mathrm{R}}{\partial a_3}=0 \quad (27)$$

Now a occurs explicitly in R and implicitly with n in the mean anomaly $(nt+a_3)$. We have, then, that

$$\frac{\partial \mathrm{R}}{\partial a_1}=\left(\frac{\partial \mathrm{R}}{\partial a_1}\right)+\frac{\partial \mathrm{R}}{dn}\frac{\partial n}{da_1}$$

or

$$\frac{\partial \mathrm{R}}{\partial a_1}=\left(\frac{\partial \mathrm{R}}{\partial a_1}\right)+t\frac{dn}{da_1}\frac{\partial \mathrm{R}}{\partial a_3} \quad . \quad . \quad . \quad . \quad (28)$$

where $\left(\frac{\partial \mathrm{R}}{\partial a_1}\right)$ means differentiation with respect to a_1, n in the mean anomaly $(nt+a_3)$ being supposed constant.

Now

$$u-e\sin u=nt+a_3$$

$$\therefore \quad r\frac{\partial u}{\partial a_1}=t\frac{dn}{da_1} \quad . \quad . \quad . \quad . \quad (29)$$

Having regard to (28) and (29), we find that equation (27) reduces to

$$a(\cos u+e)\left(\frac{\partial R}{\partial a_1}\right)+(1-e^2)\frac{\partial R}{\partial a_2}-\sin u(2-e\cos u-e^2)\frac{\partial R}{\partial a_3}=0 \quad . \quad (30)$$

or, in the usual notation,

$$a(\cos u+e)\left(\frac{\partial R}{\partial a}\right)+(1-e^2)\frac{\partial R}{\partial e}-\sin u(2-e\cos u-e^2)\frac{\partial R}{\partial \chi}=0 \quad . \quad (B)$$

This is the second relation between derivatives of the disturbing function with respect to the elliptic elements.

§ 5. If there are any more independent relations of the type (19) and (30), these can only arise through the vanishing of determinants of the forms

$$\frac{\partial(x, y, z)}{\partial(a_s, b_i, b_j)} \quad \text{and} \quad \frac{\partial(x, y, z)}{\partial(b_s, a_i, a_j)}.$$

Now it is easy to see, having regard to the values of

$$\frac{\partial(y, z)}{\partial(b_i, b_j)}, \qquad \frac{\partial(z, x)}{\partial(b_i, b_j)}, \qquad \frac{\partial(x, y)}{\partial(b_i, b_j)},$$

that the former determinant is not zero whatever the values of s, i, and j may be. Now from (24) we have

$$\frac{1}{l_3}\frac{\partial(y, z)}{\partial(a_2, a_3)}=\frac{1}{m_3}\frac{\partial(z, x)}{\partial(a_2, a_3)}=\frac{1}{n_3}\frac{\partial(x, y)}{\partial(a_2, a_3)}=\frac{\partial(\xi, \eta)}{\partial(a_2, a_3)} \quad . \quad (31)$$

and consequently

$$\frac{\partial(x, y, z)}{\partial(b_s, a_2, a_3)}=\frac{\partial(\xi, \eta)}{\partial(a_2, a_3)}\left\{l_3\frac{\partial x}{\partial b_s}+m_3\frac{\partial y}{\partial b_s}+n_3\frac{\partial z}{\partial b_s}\right\}.$$

Now, from (9), the right-hand side vanishes if $s=2$, so that

$$\frac{\partial(x, y, z)}{\partial(b_2, a_2, a_3)}=0 \quad . \quad . \quad . \quad . \quad (32)$$

There is no other determinant, of the forms considered at the beginning of this paragraph, which is zero. Thus there are three, and only three, independent relations amongst the derivatives of the disturbing function with respect to the six elliptic elements.

To derive the third, we proceed as in §§ 3, 4 and obtain the equations

$$-\frac{\partial R}{\partial b_2}=X\frac{\partial x}{\partial b_2}+Y\frac{\partial y}{\partial b_2}+Z\frac{\partial z}{\partial b_2},$$
$$-\frac{\partial R}{\partial a_2}=X\frac{\partial x}{\partial a_2}+Y\frac{dy}{\partial a_2}+Z\frac{\partial z}{\partial a_2},$$
$$-\frac{\partial R}{\partial a_3}=X\frac{\partial x}{\partial a_3}+Y\frac{\partial y}{\partial a_3}+Z\frac{\partial z}{\partial a_3},$$

from which, using (32), we obtain

$$\frac{\partial(y,z)}{\partial(a_2,a_3)}\frac{\partial \mathrm{R}}{\partial b_2}+\frac{\partial(y,z)}{\partial(a_3,b_2)}\frac{\partial \mathrm{R}}{\partial a_2}+\frac{\partial(y,z)}{\partial(b_2,a_2)}\frac{\partial \mathrm{R}}{\partial a_3}=0 \quad . \quad . \quad (33)$$

Now,

$$\begin{aligned}\frac{\partial(y,z)}{\partial(a_3,b_2)} &= l_3 r\frac{\partial r}{\partial a_3}\\ &= l_3 a^2 e\sin u\\ \frac{\partial(y,z)}{\partial(b_2,a_2)} &= -l_3 r\frac{\partial r}{\partial a_2}\\ &= l_3 a^2(\cos u-e)\\ \frac{\partial(y,z)}{\partial(a_2,a_3)} &= l_3\frac{\partial(\xi,\eta)}{\partial(a_2,a_3)}\\ &= -l_3 a^2(1-e^2)^{-\frac{1}{2}}(\cos u+e).\end{aligned}$$

Hence,

$$(\cos u+e)\frac{\partial \mathrm{R}}{\partial b_2}=\sqrt{1-e^2}\left\{e\sin u\frac{\partial \mathrm{R}}{\partial a_2}+(\cos u-e)\frac{\partial \mathrm{R}}{\partial a_3}\right\} \quad . \quad (34)$$

or, in the usual notation,

$$(\cos u+e)\frac{\partial \mathrm{R}}{\partial \omega}=\sqrt{1-e^2}\left\{e\sin u\frac{\partial \mathrm{R}}{\partial e}+(\cos u-e)\frac{\partial \mathrm{R}}{\partial \chi}\right\} \quad . \quad (\mathrm{C})$$

This is the third of the three independent relations between the derivatives of the disturbing function.

The existence of three independent relations follows from the theorem of Professor Baker's paper below; I have verified that his equations yield the results (A), (B), and (C).

§ 6. We shall now derive the following two formulæ:—

$$l_3\mathrm{X}+m_3\mathrm{Y}+n_3\mathrm{Z}=-\frac{1}{\mathrm{K}}\frac{\partial \mathrm{R}}{\partial b_3} \quad . \quad . \quad . \quad (35)$$

$$\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z=-a\left(\frac{\partial \mathrm{R}}{\partial a_1}\right) \quad . \quad . \quad . \quad (36)$$

Consider the equations

$$\begin{aligned}-\frac{\partial \mathrm{R}}{\partial b_3} &= \mathrm{X}\frac{\partial x}{\partial b_3}+\mathrm{Y}\frac{\partial y}{\partial b_3}+\mathrm{Z}\frac{\partial z}{\partial b_3},\\ -\frac{\partial \mathrm{R}}{\partial a_2} &= \mathrm{X}\frac{\partial x}{\partial a_2}+\mathrm{Y}\frac{\partial y}{\partial a_2}+\mathrm{Z}\frac{\partial z}{\partial a_2},\\ -\frac{\partial \mathrm{R}}{\partial a_3} &= \mathrm{X}\frac{\partial x}{\partial a_3}+\mathrm{Y}\frac{dy}{\partial a_3}+\mathrm{Z}\frac{\partial z}{\partial a_3},\end{aligned}$$

obtained by differentiating R with respect to b_3, a_2, and a_3.

Then we find

$$-\frac{\partial(x,y,z)}{\partial(b_3,a_2,a_3)}\mathrm{X}=\frac{\partial(y,z)}{\partial(a_2,a_3)}\frac{\partial \mathrm{R}}{\partial b_3}+\frac{\partial(y,z)}{\partial(a_3,b_3)}\frac{\partial \mathrm{R}}{\partial a_2}+\frac{\partial(y,z)}{\partial(b_3,a_2)}\frac{\partial \mathrm{R}}{\partial a_3} \quad (37)$$

with symmetrical equations in Y and Z.

Now

$$\frac{\partial(x, y, z)}{\partial(b_3, a_2, a_3)} = \frac{\partial x}{\partial b_3}\cdot\frac{\partial(y, z)}{\partial(a_2, a_3)}+\frac{\partial y}{\partial b_3}\cdot\frac{\partial(z, x)}{\partial(a_2, a_3)}+\frac{\partial z}{\partial b_3}\cdot\frac{\partial(x, y)}{\partial(a_2, a_3)}$$
$$= \mathrm{K}\frac{\partial(\xi, \eta)}{\partial(a_2, a_3)} \quad \text{from (9) and (31).}$$

Also

$$\frac{\partial(y, z)}{\partial(a_2, a_3)} = l_3\frac{\partial(\xi, \eta)}{\partial(a_2, a_3)},$$
$$\frac{\partial(y, z)}{\partial(a_3, b_3)} = -\mathrm{K}\Big(l_2\frac{\partial\xi}{\partial a_3}-l_1\frac{\partial\eta}{\partial a_3}\Big),$$
$$\frac{\partial(y, z)}{\partial(b_3, a_2)} = \mathrm{K}\Big(l_2\frac{\partial\xi}{\partial a_2}-l_1\frac{\partial\eta}{\partial a_2}\Big).$$

Hence (37) becomes

$$\mathrm{X}\frac{\partial(\xi, \eta)}{\partial(a_2, a_3)} = \Big(l_2\frac{\partial\xi}{\partial a_3}-l_1\frac{\partial\eta}{\partial a_3}\Big)\frac{\partial\mathrm{R}}{\partial a_2}-\Big(l_2\frac{\partial\xi}{\partial a_2}-l_1\frac{\partial\eta}{\partial a_2}\Big)\frac{\partial\mathrm{R}}{\partial a_3}-\frac{l_3}{\mathrm{K}}\frac{\partial(\xi, \eta)}{\partial(a_2, a_3)}\frac{\partial\mathrm{R}}{\partial b_3} \quad (38)$$

with two symmetrical equations in Y and Z.

Multiplying the three equations (38) in order by l_3, m_3, n_3, we have

$$l_3\mathrm{X}+m_3\mathrm{Y}+n_3\mathrm{Z} = -\frac{1}{\mathrm{K}}\frac{\partial\mathrm{R}}{\partial b_3} \quad . \quad . \quad . \quad (35)$$

Again, multiplying the three equations (38) in order by x, y, z we have, from (22) and remembering that $x = l_1\xi+l_2\eta$, etc.,

$$\frac{\partial(\xi, \eta)}{\partial(a_2, a_3)}(\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z) = \Big(\eta\frac{\partial\xi}{\partial a_3}-\xi\frac{\partial\eta}{\partial a_3}\Big)\frac{\partial\mathrm{R}}{\partial a_2}-\Big(\eta\frac{\partial\xi}{\partial a_2}-\xi\frac{\partial\eta}{\partial a_2}\Big)\frac{\partial\mathrm{R}}{\partial a_3} \quad (39)$$

But

$$\frac{\partial(\xi, \eta)}{\partial(a_2, a_3)} = -a^2(1-e^2)^{-\frac{1}{2}}(\cos u+e)$$
$$\eta\frac{\partial\xi}{\partial a_3}-\xi\frac{\partial\eta}{\partial a_3} = -a^2(1-e^2)^{\frac{1}{2}}$$
$$\eta\frac{\partial\xi}{\partial a_2}-\xi\frac{\partial\eta}{\partial a_2} = -a^2(1-e^2)^{-\frac{1}{2}}\sin u(2-e\cos u-e^2).$$

Hence (39) becomes

$$(\cos u+e)(\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z) = (1-e^2)\frac{\partial\mathrm{R}}{\partial a_2}-\sin u(2-e\cos u-e^2)\frac{\partial\mathrm{R}}{\partial a_3}.$$

But the right-hand side of this equation is by (30) equal to

$$-a(\cos u+e)\Big(\frac{\partial\mathrm{R}}{\partial a_1}\Big);$$

hence we have

$$\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z = -a\Big(\frac{\partial\mathrm{R}}{\partial a_1}\Big) \quad . \quad . \quad . \quad (36)$$

The equation (35) can be derived very simply, by noting that

$$-\frac{\partial \mathrm{R}}{\partial b_3} = \mathrm{X}\frac{\partial x}{\partial b_3} + \mathrm{Y}\frac{\partial y}{\partial b_3} + \mathrm{Z}\frac{\partial z}{\partial b_3}$$

and that—from (9)—

$$\frac{\partial x}{\partial b_3} = l_3\mathrm{K} \quad : \quad \frac{\partial y}{\partial b_3} = m_3\mathrm{K} \quad : \quad \frac{\partial z}{\partial b_3} = n_3\mathrm{K}.$$

The result (35) follows immediately.

§ 7. Differentiating R with respect to a_s (any one of the three a's) and b_2, b_3, and proceeding in the same way as before, we find

$$-\frac{\partial(x, y, z)}{\partial(a_s, b_2, b_3)}\mathrm{X} = \frac{\partial(y, z)}{\partial(b_2, b_3)}\frac{\partial \mathrm{R}}{\partial a_s} + \frac{\partial(y, z)}{\partial(b_3, a_s)}\frac{\partial \mathrm{R}}{\partial b_2} + \frac{\partial(y, z)}{\partial(a_s, b_2)}\frac{\partial \mathrm{R}}{\partial b_3} \quad . \quad (40)$$

with two symmetrical equations in Y and Z.

Now

$$\frac{\partial(x, y, z)}{\partial(a_s, b_2, b_3)} = \mathrm{K}\left[\frac{\partial x}{\partial a_s}(l_1\xi + l_2\eta) + \frac{\partial y}{\partial a_s}(m_1\xi + m_2\eta) + \frac{\partial z}{\partial a_s}(n_1\xi + n_2\eta)\right]$$

$$= \mathrm{K}r\frac{\partial r}{\partial a_s}.$$

Also

$$\frac{\partial(y, z)}{\partial(b_2, b_3)} = \mathrm{K}x,$$

and

$$\frac{\partial(y, z)}{\partial(b_3, a_s)} = \mathrm{K}\frac{\partial^2 x}{\partial b_2 \partial a_s},$$

and

$$\frac{\partial(y, z)}{\partial(a_s, b_2)} = l_3 r\frac{\partial r}{\partial a_s}.$$

Equation (40) then becomes

$$-\mathrm{K}r\frac{\partial r}{\partial a_s}\mathrm{X} = \mathrm{K}x\frac{\partial \mathrm{R}}{\partial a_s} + \mathrm{K}\frac{\partial^2 x}{\partial b_2 \partial a_s}\frac{\partial \mathrm{R}}{\partial b_2} + l_3 r\frac{\partial r}{\partial a_s}\frac{\partial \mathrm{R}}{\partial b_3} \quad . \quad . \quad (41)$$

Similarly

$$-\mathrm{K}r\frac{\partial r}{\partial a_s}\mathrm{Y} = \mathrm{K}y\frac{\partial \mathrm{R}}{\partial a_s} + \mathrm{K}\frac{\partial^2 y}{\partial b_2 \partial a_s}\frac{\partial \mathrm{R}}{\partial b_2} + m_3 r\frac{\partial r}{\partial a_s}\frac{\partial \mathrm{R}}{\partial b_3} \quad . \quad . \quad (42)$$

$$-\mathrm{K}r\frac{\partial r}{\partial a_s}\mathrm{Z} = \mathrm{K}z\frac{\partial \mathrm{R}}{\partial a_s} + \mathrm{K}\frac{\partial^2 z}{\partial b_2 \partial a_s}\frac{\partial \mathrm{R}}{\partial b_2} + n_3 r\frac{\partial r}{\partial a_s}\frac{\partial \mathrm{R}}{\partial b_3} \quad . \quad . \quad (43)$$

Multiply these in order by x, y, and z and add. Then, since

$$l_3x + m_3y + n_3z = 0,$$

$$\mathrm{X}x + \mathrm{Y}y + \mathrm{Z}z = -a\left(\frac{\partial \mathrm{R}}{\partial a_1}\right),$$

we have

$$\mathrm{K}ar\frac{\partial r}{\partial a_s}\left(\frac{\partial \mathrm{R}}{\partial a_1}\right) = \mathrm{K}r^2\frac{\partial \mathrm{R}}{\partial a_s} + \mathrm{K}\frac{\partial \mathrm{R}}{\partial b_2}\sum_{xyz} x\frac{\partial^2 x}{\partial b_2 \partial a_s}.$$

Now

$$\sum_{xyz} x\frac{\partial^2 x}{\partial b_2 \partial a_s} = \sum_{lmn}(l_1\xi+l_2\eta)\left(l_2\frac{\partial\xi}{\partial a_s}-l_1\frac{\partial\eta}{\partial a_s}\right)$$

$$= \eta\frac{\partial\xi}{\partial a_s}-\xi\frac{\partial\eta}{\partial a_s}.$$

Hence we have

$$ar\frac{\partial r}{\partial a_s}\left(\frac{\partial \mathrm{R}}{\partial a_1}\right) = r^2\frac{\partial \mathrm{R}}{\partial a_s}+\left(\eta\frac{\partial\xi}{\partial a_s}-\xi\frac{\partial\eta}{\partial a_s}\right)\frac{\partial \mathrm{R}}{\partial b_2} \quad . \quad . \quad (44)$$

If we put $s = 2$, this equation reduces simply to equation (30) of § 4. Putting $s = 3$, we find that

$$r\frac{\partial r}{\partial a_3} = a^2e\sin u,$$

$$\eta\frac{\partial\xi}{\partial a_3}-\xi\frac{\partial\eta}{\partial a_3} = -a^2(1-e^2)^{\frac{1}{2}},$$

and equation (44) becomes

$$ae\sin u\left(\frac{\partial \mathrm{R}}{\partial a_1}\right) = \left(\frac{r}{a}\right)^2\frac{\partial \mathrm{R}}{\partial a_3}-(1-e^2)^{\frac{1}{2}}\frac{\partial \mathrm{R}}{\partial b_2} \quad . \quad . \quad (45)$$

or, in the usual notation,

$$ae\sin u\left(\frac{\partial \mathrm{R}}{\partial a}\right) = \left(\frac{r}{a}\right)^2\frac{\partial \mathrm{R}}{\partial \chi}-(1-e^2)^{\frac{1}{2}}\frac{\partial \mathrm{R}}{\partial \omega} \quad . \quad . \quad (\mathrm{D})$$

This equation (D) is not independent of (A), (B), and (C); it is easily seen to result from the elimination of $\frac{\partial \mathrm{R}}{\partial e}$ between (B) and (C).

Put $s = 1$ in (44). Then

$$\frac{\partial \mathrm{R}}{\partial a_1} = \left(\frac{\partial \mathrm{R}}{\partial a_1}\right)+t\frac{dn}{da_1}\frac{\partial \mathrm{R}}{\partial a_3},$$

$$\frac{\partial r}{\partial a_1} = \frac{r}{a}+\frac{a^2e\sin u}{r}t\frac{dn}{da_1},$$

and

$$\eta\frac{\partial\xi}{\partial a_1}-\xi\frac{\partial\eta}{\partial a_1} = -a^2(1-e^2)^{\frac{1}{2}}t\frac{dn}{da_1}.$$

We have, as when $s = 3$, the equation (45).

Multiplying the equations (41), (42), (43) in order by l_3, m_3, n_3 and remembering that

$$l_3\mathrm{X}+m_3\mathrm{Y}+n_3\mathrm{Z} = -\frac{1}{\mathrm{K}}\frac{\partial \mathrm{R}}{\partial b_3},$$

we obtain

$$l_3\frac{\partial^2 x}{\partial b_2\partial a_s}+m_3\frac{\partial^2 y}{\partial b_2\partial a_s}+n_3\frac{\partial^2 z}{\partial b_2\partial a_s} = 0.$$

Now l_3, m_3, n_3 are independent of b_2, so that the previous equation becomes

$$\frac{\partial}{\partial b_2}\left\{l_3\frac{\partial x}{\partial a_s}+m_3\frac{\partial y}{\partial a_s}+n_3\frac{\partial z}{\partial a_s}\right\}=0$$

or

$$\frac{\partial^2}{\partial b_2 \partial a_s}(l_3x+m_3y+n_3z)=0,$$

that is, we have, from (22), an identity.

Putting $s=2$ in (44), we obtain, after some reduction,

$$a(\cos u-e)\left(\frac{\partial \mathrm{R}}{\partial a}\right)+\left(\frac{r}{a}\right)^2\frac{\partial \mathrm{R}}{\partial a_2}-(1-e^2)^{-\frac{1}{2}}\sin u(2-e\cos u-e^2)\frac{\partial \mathrm{R}}{\partial b_2}=0 \quad (46)$$

or, in the usual notation,

$$a(\cos u-e)\left(\frac{\partial \mathrm{R}}{\partial a}\right)+\left(\frac{r}{a}\right)^2\frac{\partial \mathrm{R}}{\partial e}-(1-e^2)^{-\frac{1}{2}}\sin u(2-e\cos u-e^2)\frac{\partial \mathrm{R}}{\partial \omega}=0 \quad (\mathrm{E})$$

Equation (E) is also seen to be the result of eliminating $\dfrac{\partial \mathrm{R}}{\partial \chi}$ between (C) and (D).

§ 8. The equations of planetary motion can be derived in various ways. Lagrange, for example, based his analytical method on the equations (5) and (6). In the subsequent pages, I utilise these equations, together with the relations (A), (B), and (C), in obtaining the six well-known equations of planetary motion.

We rewrite them in the form

$$\sum_{s=1}^{3}\frac{\partial x}{\partial a_s}\cdot\frac{da_s}{dt}+\sum_{s=1}^{3}\frac{\partial x}{\partial b_s}\frac{db_s}{dt}=0 \quad . \quad . \quad . \quad (47)$$

with two similar equations in y and z, and

$$\sum_{s=1}^{3}\frac{\partial x'}{\partial a_s}\frac{da_s}{dt}+\sum_{s=1}^{3}\frac{\partial x'}{\partial b_s}\frac{db_s}{dt}=\frac{\partial \mathrm{R}}{\partial x} \quad . \quad . \quad . \quad (48)$$

with two similar equations in y' and z'.

Multiply the three equations (47) in order by x, y, and z and add. Then, since

$$r^2=x^2+y^2+z^2\,; \qquad r\frac{\partial r}{\partial a_s}=x\frac{\partial x}{\partial a_s}+y\frac{\partial y}{\partial a_s}+z\frac{\partial z}{\partial a_s},$$

we have

$$\sum_{s=1}^{3}\frac{\partial r}{\partial a_s}\frac{da_s}{dt}+\sum_{s=1}^{3}\frac{\partial r}{\partial b_s}\frac{db_s}{dt}=0.$$

But r is a function of a_1, a_2, a_3 and t only, so that

$$\frac{\partial r}{\partial b_s}=0 \qquad (s=1,\ 2,\ 3)$$

Hence
$$\frac{\partial r}{\partial a_1}\frac{da_1}{dt}+\frac{\partial r}{\partial a_2}\frac{da_2}{dt}+\frac{\partial r}{\partial a_3}\frac{da_3}{dt}=0 \quad . \quad . \quad . \quad (49)$$

Multiply the three equations (48) in order by x', y', and z' and add. Denote the velocity of the planet by V: then $V^2=x'^2+y'^2+z'^2$. Then we have, since for the instantaneous orbit $x'=\frac{\partial x}{\partial t}$, etc.

$$\frac{1}{2}\sum_{s=1}^{3}\frac{\partial(V^2)}{\partial a_s}\frac{da_s}{dt}+\frac{1}{2}\sum_{s=1}^{3}\frac{\partial(V^2)}{\partial b_s}\frac{db_s}{dt}=\frac{\partial R}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial R}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial t} \quad . \quad (50)$$

Now
$$V^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right),$$

so that
$$\frac{\partial(V^2)}{\partial b_s}=0.$$

Also
$$\frac{\partial(V^2)}{\partial a_1}=-\frac{2\mu}{r^2}\frac{\partial r}{\partial a_1}+\frac{\mu}{a^2},$$
$$\frac{\partial(V^2)}{\partial a_2}=-\frac{2\mu}{r^2}\frac{\partial r}{\partial a_2}$$
$$\frac{\partial(V)^2}{\partial a_3}=-\frac{2\mu}{r^2}\frac{\partial r}{\partial a_3}.$$

Also, if
$$\frac{\partial R(t,\,x,\,y,\,z)}{\partial t}$$

denotes differentiation with respect to t only in so far as t is introduced into R by x, y, and z, then
$$\frac{\partial R}{\partial t}=\frac{\partial R}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial R}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial t}.$$

Now t is introduced into R by x, y, and z in the form $l\equiv nt+\chi$, and consequently
$$\frac{\partial R}{\partial t}=n\frac{\partial R}{\partial \chi}.$$

Hence equation (50) becomes
$$-\frac{\mu}{r^2}\sum_{s=1}^{3}\frac{\partial r}{\partial a_s}\frac{da_s}{dt}+\frac{\mu}{2a^2}\frac{da}{dt}=n\frac{\partial R}{\partial \chi},$$

and having regard to (49) and the equation for μ, viz. $\mu=n^2a^3$, we obtain the first of the well-known equations of motion
$$\frac{da}{dt}=\frac{2}{na}\frac{\partial R}{\partial \chi} \quad . \quad . \quad . \quad . \quad (51)$$

§ 9. Now multiply the equations (47) in order by x', y', z' and the equations (48) in order by x, y, z and add.

In the sum, the coefficient of $\frac{da_1}{dt}$, for example, is

$$\frac{\partial}{\partial a_1}(xx'+yy'+zz'), \quad \text{that is,} \quad \frac{\partial}{\partial a_1}\left(r\frac{\partial r}{\partial t}\right).$$

We thus have

$$\sum_{s=1}^{3}\frac{\partial}{\partial a_s}\left(r\frac{\partial r}{\partial t}\right)\frac{da_s}{dt}+\sum_{s=1}^{3}\frac{\partial}{\partial b_s}\left(r\frac{dr}{dt}\right)\frac{db_s}{dt} = x\frac{\partial \mathrm{R}}{\partial x}+y\frac{\partial \mathrm{R}}{\partial y}+z\frac{\partial \mathrm{R}}{\partial z}.$$

As before,

$$\frac{\partial}{\partial b_s}\left(r\frac{\partial r}{\partial t}\right) = 0.$$

Also from (36)

$$x\frac{\partial \mathrm{R}}{\partial x}+y\frac{\partial \mathrm{R}}{\partial y}+z\frac{\partial \mathrm{R}}{\partial z} = a\left(\frac{\partial \mathrm{R}}{\partial a}\right).$$

Hence

$$\sum_{s=1}^{3}\frac{\partial}{\partial a_s}\left(r\frac{\partial r}{\partial t}\right)\frac{da_s}{dt} = a\left(\frac{\partial \mathrm{R}}{\partial a}\right) \quad . \quad . \quad . \quad (52)$$

This equation can be put in another form by making use of (49). It is

$$\sum_{s=1}^{3}\frac{\partial}{\partial a_s}\left(\frac{\partial r}{\partial t}\right)\frac{da_s}{dt} = \frac{a}{r}\left(\frac{\partial \mathrm{R}}{\partial a}\right) \quad . \quad . \quad . \quad (53)$$

§ 10. Multiply the equations (47) in order by l_3, m_3, n_3 and add. Then there results

$$\sum_{s=1}^{3}\left(l_3\frac{\partial x}{\partial a_s}+m_3\frac{\partial y}{\partial a_s}+n_3\frac{\partial z}{\partial a_s}\right)\frac{da_s}{dt}+\sum_{s=1}^{3}\left(l_3\frac{\partial x}{\partial b_s}+m_3\frac{\partial y}{\partial b_s}+n_3\frac{\partial z}{\partial b_s}\right)\frac{db_s}{dt} = 0 \quad (54)$$

Now l_3, m_3, n_3 are independent of a_1, a_2, a_3, so that

$$l_3\frac{\partial x}{\partial a_s}+m_3\frac{\partial y}{\partial a_s}+n_3\frac{\partial z}{\partial a_s} = \frac{\partial}{\partial a_s}(l_3x+m_3y+n_3z) = 0 \quad \text{by (22).}$$

Hence (54) reduces to

$$\sum_{s=1}^{3}\left(l_3\frac{\partial x}{\partial b_s}+m_3\frac{\partial y}{\partial b_s}+n_3\frac{\partial z}{\partial b_s}\right)\frac{db_s}{dt} = 0 \quad . \quad . \quad . \quad (55)$$

Now using (8) and (9) we find at once

$$l_3\frac{\partial x}{\partial b_1}+m_3\frac{\partial y}{\partial b_1}+n_3\frac{\partial z}{\partial b_1} = -(n_2\xi-n_1\eta),$$

$$l_3\frac{\partial x}{\partial b_2}+m_3\frac{\partial y}{\partial b_2}+n_3\frac{\partial z}{\partial b_2} = 0,$$

$$l_3\frac{\partial x}{\partial b_3}+m_3\frac{\partial y}{\partial b_3}+n_3\frac{\partial z}{\partial b_3} = \mathrm{K},$$

so that (55) becomes

$$-(n_2\xi-n_1\eta)\frac{db_1}{dt}+\mathrm{K}\frac{db_3}{dt}=0 \quad . \quad . \quad . \quad (56)$$

Now multiply the equations (48) by l_3, m_3, n_3 and add. Let us write

$$\mathrm{K}'=\xi' \sin b_2+\eta' \cos b_2 \quad . \quad . \quad . \quad (57)$$

and we have at once

$$-(n_2\xi'-n_1\eta')\frac{db_1}{dt}+\mathrm{K}'\frac{db_3}{dt}=l_3\frac{\partial \mathrm{R}}{\partial x}+m_3\frac{\partial \mathrm{R}}{\partial y}+n_3\frac{\partial \mathrm{R}}{\partial z}=\frac{1}{\mathrm{K}}\frac{\partial \mathrm{R}}{\partial b_3} \quad . \quad (58)$$

by utilising (35).

Eliminate $\frac{db_3}{dt}$ between (56) and (58). Then we obtain

$$[\mathrm{K}'(n_2\xi-n_1\eta)-\mathrm{K}(n_2\xi'-n_1\eta')]\frac{db_1}{dt}=\frac{\partial \mathrm{R}}{\partial b_3}.$$

The coefficient of $\frac{db_1}{dt}$ is easily found to be

$$(\xi\eta'-\xi'\eta)(n_2 \cos b_2+n_1 \sin b_2),$$

or, from (7),

$$\sin b_3(\xi\eta'-\xi'\eta).$$

But in elliptic motion,

$$\xi\eta'-\xi'\eta=na^2(1-e^2)^{\frac{1}{2}}\equiv h.$$

Hence we have

$$\frac{db_1}{dt}=\frac{1}{h \sin b_3}\frac{\partial \mathrm{R}}{\partial b_3} \quad . \quad . \quad . \quad . \quad (59)$$

or, in the usual notation,

$$\frac{d\theta}{dt}=\frac{1}{na^2(1-e^2)^{\frac{1}{2}} \sin i}\frac{\partial \mathrm{R}}{di} \quad . \quad . \quad . \quad (60)$$

which is the well-known equation.

§ 11. From (56) we deduce the equation for $\frac{db_3}{dt}$. We have

$$\mathrm{K}\frac{db_3}{dt}=(n_2\xi-n_1\eta)\frac{db_1}{dt}$$

$$=\frac{(n_2\xi-n_1\eta)}{h \sin b_3}\frac{\partial \mathrm{R}}{\partial b_3} \quad \text{from (59).}$$

But from (17) and (18)—in essence, this is equivalent to using (A)—

$$\mathrm{K}\frac{\partial \mathrm{R}}{\partial b_1}-\mathrm{K}n_3\frac{\partial \mathrm{R}}{\partial b_2}+(n_2\xi-n_1\eta)\frac{\partial \mathrm{R}}{\partial b_3}=0.$$

Consequently,

$$\frac{db_3}{dt}=-\frac{1}{h \sin b_3}\frac{\partial \mathrm{R}}{\partial b_1}+\frac{\cot b_3}{h}\frac{\partial \mathrm{R}}{\partial b_2} \quad . \quad . \quad . \quad (61)$$

or, in the usual notation,

$$\frac{di}{dt} = -\frac{1}{na^2(1-e^2)^{\frac{1}{2}}\sin i}\frac{\partial \mathrm{R}}{\partial \theta} + \frac{\cot i}{na^2(1-e^2)^{\frac{1}{2}}}\frac{\partial \mathrm{R}}{\partial \omega} \quad . \quad . \quad (62)$$

which is again one of the well-known equations of planetary motion.

§ 12. Multiply the equations (48) in order by $\frac{\partial x}{\partial b_1}, \frac{\partial y}{\partial b_1}, \frac{\partial z}{\partial b_1}$, and add. Then we have

$$\sum_{s=1}^{3}\left\{\sum_{xyz}\frac{\partial x'}{\partial a_s}\frac{\partial x}{\partial b_1}\right\}\frac{da_s}{dt} + \sum_{s=1}^{3}\left\{\sum_{xyz}\frac{\partial x'}{\partial b_s}\cdot\frac{\partial x}{\partial b_1}\right\}\frac{db_s}{dt} = \frac{\partial \mathrm{R}'}{\partial b_1}.$$

But from (9),

$$\frac{\partial x}{\partial b_1} = -y \quad : \quad \frac{\partial y}{\partial b_1} = x \quad : \quad \frac{\partial z}{\partial b_1} = 0,$$

so that we have

$$\sum_{s=1}^{3}\left(x\frac{\partial y'}{\partial a_s} - y\frac{\partial x'}{\partial a_s}\right)\frac{da_s}{dt} + \sum_{s=1}^{3}\left(x\frac{\partial y'}{\partial b_s} - y\frac{\partial x'}{\partial b_s}\right)\frac{db_s}{dt} = \frac{\partial \mathrm{R}}{\partial b_1} \quad . \quad . \quad (63)$$

Now multiply the equations (47) in order by $\frac{\partial x'}{\partial b_1}, \frac{\partial y'}{\partial b_1}, \frac{\partial z'}{\partial b_1}$ and add. From (9) we have

$$\frac{\partial x'}{\partial b_1} = -y' \quad : \quad \frac{\partial y'}{\partial b_1} = x' \quad : \quad \frac{\partial z'}{\partial b_1} = 0.$$

We obtain as before

$$\sum_{s=1}^{3}\left(x'\frac{\partial y}{\partial a_s} - y'\frac{\partial x}{\partial a_s}\right)\frac{da_s}{dt} + \sum_{s=1}^{3}\left(x'\frac{\partial y}{\partial b_s} - y'\frac{\partial x}{\partial b_s}\right)\frac{db_s}{dt} = 0 \quad . \quad (64)$$

Subtract (64) from (63) and there results

$$\sum_{s=1}^{3}\frac{\partial}{\partial a_s}(xy'-x'y)\frac{da_s}{dt} + \sum_{s=1}^{3}\frac{\partial}{\partial b_s}(xy'-x'y)\frac{db_s}{dt} = \frac{\partial \mathrm{R}}{\partial b_1}.$$

But

$$\begin{aligned} xy'-x'y &= n_3(\xi\eta'-\xi'\eta) \\ &= h\cos b_3. \end{aligned}$$

Writing

$$\mathrm{F} = h\cos b_3 = na^2(1-e^2)^{\frac{1}{2}}\cos b_3 = \mu^{\frac{1}{2}}a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}\cos b_3$$

we obtain simply

$$\frac{\partial \mathrm{F}}{\partial a_1}\frac{da_1}{dt} + \frac{\partial \mathrm{F}}{\partial a_2}\frac{da_2}{dt} + \frac{\partial \mathrm{F}}{\partial b_3}\frac{db_3}{dt} = \frac{\partial \mathrm{R}}{\partial b_1},$$

which leads to the equation

$$\frac{1}{2a}\frac{da_1}{dt} - \frac{e}{1-e^2}\frac{da_2}{dt} - \tan b_3\frac{db_3}{dt} = \frac{1}{\mathrm{F}}\frac{\partial \mathrm{R}}{\partial b_1} \quad . \quad . \quad (65)$$

But from (51) and (61)

$$\frac{da_1}{dt} = \frac{2}{na}\frac{\partial \mathrm{R}}{\partial a_3},$$

and

$$\tan b_3 \frac{db_3}{dt} = -\frac{1}{\mathrm{F}}\frac{\partial \mathrm{R}}{\partial b_1} + \frac{\cos b_3}{\mathrm{F}}\frac{\partial \mathrm{R}}{\partial b_2}.$$

Consequently (65) reduces to

$$\frac{da_2}{dt} = \frac{(1-e^2)}{na^2e}\frac{\partial \mathrm{R}}{\partial a_3} - \frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\frac{\partial \mathrm{R}}{\partial b_2} \quad . \quad . \quad . \quad (66)$$

or, in the usual notation,

$$\frac{de}{dt} = \frac{(1-e^2)}{na^2e}\frac{\partial \mathrm{R}}{\partial \chi} - \frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\frac{\partial \mathrm{R}}{\partial \omega} \quad . \quad . \quad . \quad (67)$$

which is another of the six equations of planetary motion.

§ 13. Multiply the equations (47) in order by l_1, m_1, n_1 and add. Then

$$\sum_{s=1}^{3}\frac{\partial}{\partial a_s}(l_1x+m_1y+n_1z)\frac{da_s}{dt} = -\sum_{s=1}^{3}\left(l_1\frac{\partial x}{\partial b_s}+m_1\frac{\partial y}{\partial b_s}+n_1\frac{\partial z}{\partial b_s}\right)\frac{db_s}{dt}$$

$$= \sum_{s=1}^{3}\left(x\frac{\partial l_1}{\partial b_s}+y\frac{\partial m_1}{\partial b_s}+z\frac{\partial n_1}{\partial b_s}\right)\frac{db_s}{dt},$$

since

$$\frac{\partial}{\partial b_s}(l_1x+m_1y+n_1z) = \frac{\partial \xi}{\partial b_s} = 0.$$

Now

$$x\frac{\partial l_1}{\partial b_1}+y\frac{\partial m_1}{\partial b_1}+z\frac{\partial n_1}{\partial b_1} = -m_1x+l_1y = n_3\eta,$$

$$x\frac{\partial l_1}{\partial b_2}+y\frac{\partial m_1}{\partial b_2}+z\frac{\partial n_1}{\partial b_2} = l_2x+m_2y+n_2z = \eta,$$

$$x\frac{\partial l_1}{\partial b_3}+y\frac{\partial m_1}{\partial b_3}+z\frac{\partial n_1}{\partial b_3} = \sin b_2(l_3x+m_3y+n_3z) = 0.$$

Hence we obtain

$$\sum_{s=1}^{3}\frac{\partial \xi}{\partial a_s}\frac{da_s}{dt} = n_3\eta\frac{db_1}{dt}+\eta\frac{db_2}{dt} \quad . \quad . \quad . \quad (68)$$

Multiply the equations (47) in order by l_2, m_2, n_2 and add. We have, similarly,

$$\sum_{s=1}^{3}\frac{\partial \eta}{\partial a_s}\frac{da_s}{dt} = -n_3\xi\frac{db_1}{dt}-\xi\frac{db_2}{dt} \quad . \quad . \quad . \quad (69)$$

Multiply (68) and (69) by η' and $-\xi'$ respectively and add. Then we have

$$\sum_{s=1}^{3}\left(\eta'\frac{\partial \xi}{\partial a_s}-\xi'\frac{\partial \eta}{\partial a_s}\right)\frac{da_s}{dt} = r\frac{dr}{dt}\left(n_3\frac{db_1}{dt}+\frac{db_2}{dt}\right) \quad . \quad . \quad (70)$$

From (12) and (13) we derive the following :

$$\eta'\frac{\partial\xi}{\partial a_1}-\xi'\frac{\partial\eta}{\partial a_1}=na(1-e^2)^{\frac{1}{2}},$$

$$\eta'\frac{\partial\xi}{\partial a_2}-\xi'\frac{\partial\eta}{\partial a_2}=-na^2(1-e^2)^{-\frac{1}{2}}(\cos u+e),$$

$$\eta'\frac{\partial\xi}{\partial a_3}-\xi'\frac{\partial\eta}{\partial a_3}=0,$$

$$r\frac{dr}{dt}=na^2e\sin u.$$

Hence (70) becomes

$$na(1-e^2)^{\frac{1}{2}}\frac{da_1}{dt}-na^2(1-e^2)^{-\frac{1}{2}}(\cos u+e)\frac{da_2}{dt}=na^2e\sin u\left(n_3\frac{db_1}{dt}+\frac{db_2}{dt}\right).$$

The left-hand side of this equation becomes, when the values of $\frac{da_1}{dt}$ and $\frac{da_2}{dt}$ from (51) and (66) are inserted,

$$-\frac{(1-e^2)^{\frac{1}{2}}}{e}(\cos u-e)\frac{\partial R}{\partial a_3}+\frac{(\cos u+e)}{e}\frac{\partial R}{\partial b_2}.$$

But from (C) we have

$$(1-e^2)^{\frac{1}{2}}(\cos u-e)\frac{\partial R}{\partial a_3}-(\cos u+e)\frac{\partial R}{\partial b_2}=-e(1-e^2)^{\frac{1}{2}}\sin u\frac{\partial R}{\partial a_2}.$$

Hence we obtain the equation

$$na^2e\left(n_3\frac{db_1}{dt}+\frac{db_2}{dt}\right)=(1-e^2)^{\frac{1}{2}}\frac{\partial R}{\partial a_2} \quad . \quad . \quad . \quad (71)$$

In this equation, insert the value of $\frac{db_1}{dt}$ from (59). The result is

$$\frac{db_2}{dt}=\frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\frac{\partial R}{\partial a_2}-\frac{\cot b_3}{na^2(1-e^2)^{\frac{1}{2}}}\frac{\partial R}{\partial b_3} \quad . \quad . \quad . \quad (72)$$

or, in the usual notation,

$$\frac{d\omega}{dt}=\frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\frac{\partial R}{\partial e}-\frac{\cot i}{na^2(1-e^2)^{\frac{1}{2}}}\frac{\partial R}{\partial i} \quad . \quad . \quad . \quad (73)$$

which is the fifth equation of planetary motion.

§ 14. Eliminate $\frac{da_2}{dt}$ between equations (68) and (69). We obtain

$$\frac{\partial(\xi,\eta)}{\partial(a_1,a_2)}\frac{da_1}{dt}+\frac{\partial(\xi,\eta)}{\partial(a_3,a_2)}\frac{da_3}{dt}=\left(n_3\frac{db_1}{dt}+\frac{db_2}{dt}\right)\left(\xi\frac{\partial\xi}{\partial a_2}+\eta\frac{\partial\eta}{\partial a_2}\right)$$

$$=r\frac{\partial r}{\partial a_2}\frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\frac{\partial R}{\partial a_2} \quad \text{from (71).}$$

Now

$$\frac{\partial(\xi, \eta)}{\partial(a_1, a_2)} = a(1-e^2)^{-\frac{1}{2}} \sin u(2-e\cos u-e^2)+a^2(1-e^2)^{-\frac{1}{2}}(\cos u+e)\left[t\frac{dn}{da}\right]$$

$$\frac{\partial(\xi, \eta)}{\partial(a_3, a_2)} = a^2(1-e^2)^{-\frac{1}{2}}(\cos u+e)$$

$$r\frac{\partial r}{\partial a_2} = -a^2(\cos u-e).$$

Also

$$\frac{da_1}{dt} = \frac{2}{na}\frac{\partial \mathrm{R}}{\partial a_3} \quad \text{from (51)}$$

and

$$\left[t\frac{dn}{da}\right]\frac{\partial \mathrm{R}}{\partial a_3} = \frac{\partial \mathrm{R}}{\partial a_1} - \left(\frac{\partial \mathrm{R}}{\partial a_1}\right) \quad \text{from (28).}$$

We then obtain

$$\frac{2}{n}\sin u(2-e\cos u-e^2)\frac{\partial \mathrm{R}}{\partial a_3} - \frac{2a}{n}(\cos u+e)\left(\frac{\partial \mathrm{R}}{\partial a_1}\right) + \frac{2a}{n}(\cos u+e)\frac{\partial \mathrm{R}}{\partial a_1}$$
$$+ \frac{(1-e^2)(\cos u-e)}{ne}\frac{\partial \mathrm{R}}{\partial a_2} + a^2(\cos u+e)\frac{da_3}{dt} = 0.$$

But, from (B),

$$\frac{2}{n}\sin u(2-e\cos u-e^2)\frac{\partial \mathrm{R}}{\partial a_3} - \frac{2a}{n}(\cos u+e)\left(\frac{\partial \mathrm{R}}{\partial a_1}\right) = \frac{2}{n}(1-e^2)\frac{\partial \mathrm{R}}{\partial a_2}.$$

Hence

$$\frac{da_3}{dt} = -\frac{2}{na}\frac{\partial \mathrm{R}}{\partial a_1} - \frac{(1-e^2)}{na^2e}\frac{\partial \mathrm{R}}{\partial a_2} \quad . \quad . \quad . \quad (74)$$

or, in the usual notation,

$$\frac{d\chi}{dt} = -\frac{2}{na}\frac{\partial \mathrm{R}}{\partial a} - \frac{(1-e^2)}{na^2e}\frac{\partial \mathrm{R}}{\partial e} \quad . \quad . \quad . \quad (75)$$

the last of the six equations of planetary motion.

Two Elementary Formulæ for Planetary Motion.

By H. F. Baker, Sc.D., F.R.S.

§ 1. There are some well-known formulæ (Tisserand, *Mécanique Céleste*, i., 1889, p. 433; Brown's *Lunar Theory*, 1896, pp. 63, 64) by which the partial derivatives of the disturbing function, six in number, are expressible in terms of three component accelerations. By elimination of these accelerations three relations connecting the partial derivatives are determinable. I have been in the habit of giving these, as having a certain pedagogic value, in lecture, supposing that they must occur somewhere in the literature. Some time ago, however, Dr. W. M. Smart showed to me some work (printed above) in which formulæ,

which he has now verified to lead to the same results, occur from a quite different point of view. Perhaps, therefore, I may be allowed to present the matter as it had occurred to me, stated, in terms of Delaunay's variables, as an elementary kinematical result.

I prove the following result: A particle is moving in any manner. For any position, P, of the particle, an ellipse is described, with focus at an arbitrary fixed point, S, in the plane joining S to the tangent line of the path at P, to touch this path at P, with curvature at P equal to $\mu p/v^2r^3$; where $r = \mathrm{SP}$, v is the velocity of the particle at P, p is the perpendicular from S to the tangent line at P, and μ is any fixed constant. Take a fixed plane σ, passing through S, and a fixed line, λ, through S, in this plane; let the line of intersection of the plane of the ellipse with the plane σ make an angle h with the line λ; let the inclination of the plane of the ellipse with the plane σ be denoted by i; let the angle between the (directed) major axis of the ellipse and the line of intersection of the two planes be g. Further, let e be the eccentricity of the ellipse, $2a$ its major axis, u the eccentric anomaly of P in this ellipse, and w the (true) focal anomaly, both measured from the major axis. We have defined the two Delaunay variables g, h; define further the variable l by $l = u - e\sin u$, and the variables L, G, H by means of

$$\mathrm{L} = (\mu a)^{\frac{1}{2}}, \quad \mathrm{G} = (1-e^2)^{\frac{1}{2}}\mathrm{L}, \quad \mathrm{H} = \mathrm{G}\cos i,$$

where, as stated, μ is arbitrary. It is known that l, g, h and L, G, H form a canonical system. Put also

$$n = \mu^2\mathrm{L}^{-3}, \quad \text{so that} \quad n^2 = \mu^4(\mu a)^{-3} = \mu a^{-3}.$$

Then we have the following formulæ, where t is the time in the actual orbit,

$$\begin{aligned}
\frac{dl}{dt} - n &= a\frac{d\mathrm{L}}{dt} + k\frac{d\mathrm{G}}{dt}, \\
\frac{dg}{dt} &= k\frac{d\mathrm{L}}{dt} + b\frac{d\mathrm{G}}{dt} + f\frac{d\mathrm{H}}{dt}, \\
\frac{dh}{dt} &= f\frac{d\mathrm{G}}{dt} + c\frac{d\mathrm{H}}{dt},
\end{aligned}$$

in which the coefficients a, b, c, f, o, k enter symmetrically, these being in fact given by

$$\begin{aligned}
a &= \frac{\mu r\mathrm{G}}{\mathrm{L}^4}\,\frac{\cos w + e(\cos^2 w - 2)}{e^2\sin w}, \\
b &= \mathrm{G}^{-1}\left[\frac{\cos w + 2e + e^2\cos w}{e^2\sin w} - \cot^2 i\,.\tan(g+w)\right], \\
c &= -\mathrm{G}^{-1}\frac{\tan(g+w)}{\sin^2 i}, \qquad f = \mathrm{G}^{-1}\frac{\cos i\,.\tan(g+w)}{\sin^2 i}, \\
\mathrm{o} &= \mathrm{o}, \qquad\qquad k = -\frac{\mathrm{G}^2}{e^2\mathrm{L}^3}\cot w.
\end{aligned}$$

Further, if we denote the accelerations of the particle, in the three

rectangular directions respectively, in SP, at right angles to SP in the plane of the ellipse, and at right angles to the plane of the ellipse, by

$$-\frac{\mu}{r^2}+\mathrm{R},\ \mathrm{T},\ \mathrm{K},$$

then

$$er\sin w\,.\,\mathrm{R}=\frac{\mu r\mathrm{G}}{\mathrm{L}^3}\frac{d\mathrm{L}}{dt}-(1+e\cos w)\frac{d\mathrm{G}}{dt},$$

$$r\mathrm{T}=\frac{d\mathrm{G}}{dt},$$

$$r\sin i\cos(g+w)\mathrm{K}=\cos i\frac{d\mathrm{G}}{dt}-\frac{d\mathrm{H}}{dt}.$$

In indicating an elementary verification of these results, it may be permitted to use familiar notations for elliptic motion about a focus without individual explanation in detail.

The curvature of an ellipse of centre C is κ, given by

$$\kappa=\frac{\mathrm{PF}}{\mathrm{CD}^2}=\frac{ab}{(\mathrm{CD})^3}=\frac{a}{(\mathrm{CD})^2}\cdot\frac{b}{\mathrm{CD}}=\frac{a}{rr'}\cdot\frac{b}{(rr')^{\frac{1}{2}}}=\frac{a}{rr'}\cdot\frac{p}{r},$$

so that

$$\frac{p}{\kappa r^3}=\frac{r'}{ar}=\frac{2a-r}{ar}=\frac{2}{r}-\frac{1}{a};$$

thus the formula of the enunciation, $\kappa=\mu p/v^2r^3$, gives

$$\frac{v^2}{\mu}=\frac{2}{r}-\frac{1}{a},$$

as for a particle moving in the ellipse (as this moves) with an acceleration μr^{-2} in PS.

Again

$$p^2v^2=\mu\frac{r'}{ar}p^2=\frac{\mu}{ar}r'\,.\,r^2\frac{b^2}{rr'}=\mu\frac{b^2}{a}=\mu a(1-e^2)=\mathrm{G}^2;$$

thus G is the angular momentum of the particle about an axis through S at right angles to the plane of the ellipse. This also gives $p^2v^2=n^2a^2b^2$ or $pv=nab$.

As the ellipse touches the path, the value of dr/dt, or say $\dot{r}$, in the actual path, is the same as in the ellipse. Thus

$$\dot{r}=v\sin\mathrm{SPG}=\frac{nae}{(1-e^2)^{\frac{1}{2}}}\sin w.$$

Also we have

$$\frac{r}{a}\cos w=\cos u-e,\qquad\frac{r}{a}\frac{\sin w}{(1-e^2)^{\frac{1}{2}}}=\sin u,$$

and

$$r\dot{r}=na^2e\sin u.$$

Again, from $r=a-ae\cos u$, we get

$$\dot{u}=\left(\dot{r}-\dot{a}\frac{r}{a}+a\dot{e}\cos u\right)/ae\sin u,$$

while $l = u - e \sin u$ thence gives, with some reduction,

$$\dot{l} = n + (1-e^2)^{\frac{1}{2}} \frac{\dot{e}}{e} \cot w - \frac{\dot{a}(1-e^2)^{\frac{1}{2}}}{ae \sin w} \frac{r}{a}.$$

If we evaluate $\dot{e}$ and $\dot{a}$ by differentiating

$$(1-e^2)^{\frac{1}{2}} = \mathrm{G}/\mathrm{L}, \qquad a = \mathrm{L}^2/\mu,$$

it is at once found that the form for $\dot{l}$ is equivalent to that put down in terms of $d\mathrm{L}/dt$ and $d\mathrm{G}/dt$.

We have next to obtain $\dot{h}$ and $\dot{g}$. For this consider the diagram annexed, for the angular velocities of the directions (*cf.* Brown's *Lunar Theory*, p. 55), putting, for brevity, $v = g + w$. Here the fixed plane, σ, is that marked as containing the line λ, and the angular velocity di/dt; the plane 12 is the plane of the ellipse, the direction S2 being that to the particle. The fundamental angular velocities are $\dot{h}$ at right angles to the plane σ, di/dt about the line of intersection of σ with the plane of the ellipse, and $\dot{v}$ normal to the plane 12. If we resolve $\dot{h}$ and di/dt we obtain for the angular velocities of the axes 1, 2, 3 about themselves, respectively,

$$\mathrm{I}_1 = \frac{di}{dt} \sin v - \frac{dh}{dt} \sin i \cos v,$$

$$\mathrm{I}_2 = \frac{di}{dt} \cos v + \frac{dh}{dt} \sin i \sin v,$$

$$\mathrm{I}_3 = \dot{v} + \frac{dh}{dt} \cos i.$$

But, as the plane 12 is the plane of the ellipse, containing the tangent direction of the motion of the particle, we have $\mathrm{I}_1 = 0$. Further, $r^2\mathrm{I}_3$ is the angular momentum, or G. Thus from the first and third of these equations we deduce

$$\frac{dh}{dt} = \frac{di}{dt} \tan (g+w) \operatorname{cosec} i,$$

$$\frac{dg}{dt} = \frac{\mathrm{G}}{r^2} - \frac{dw}{dt} - \frac{dh}{dt} \cos i.$$

The former of these, by $\cos i = \mathrm{H}/\mathrm{G}$, gives the formula to be verified for dh/dt. The latter, utilising

$$\frac{a(1-e^2)}{r} = 1 + e \cos w,$$

with the values of $\dot{a}$, $\dot{e}$, $\dot{r}$ previously employed, to obtain

$$\dot{w} = \frac{\mathrm{G}}{r^2} - k\frac{d\mathrm{L}}{dt} - \mathrm{G}^{-1} \frac{1-e^2}{e^2 \sin w} \left(\frac{2ae}{r} + \cos w \right) \frac{d\mathrm{G}}{dt},$$

gives the required value for dg/dt.

The second formula of those for I_1, I_2, I_3 gives

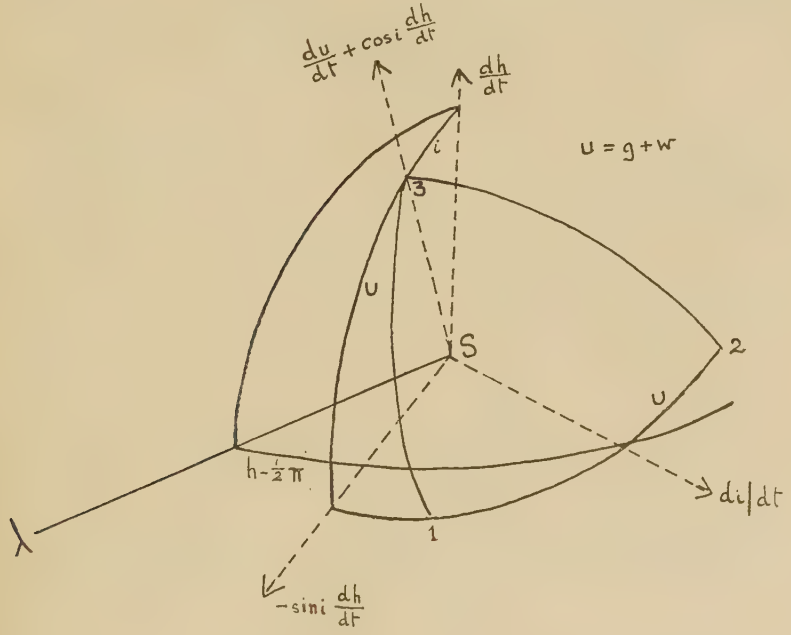
$$\mathrm{I}_2 = \operatorname{cosec} v \sin i \, dh/dt.$$

Whence, the co-ordinates of the particle being (0, r, 0), referred to the axes 1, 2, 3, the components of its velocity are

$$U = -rI_3 = -\frac{G}{r}, \qquad V = \dot{r}, \qquad W = rI_1 = 0,$$

and the accelerations of the particle, namely,

$$\dot{U} - VI_3 + WI_2, \qquad \dot{V} + UI_3, \qquad \dot{W} - UI_2,$$



may be denoted, respectively, by

$$-T, \qquad -\frac{\mu}{r^2}+R, \qquad K.$$

Thence we find

$$K = -U\frac{\dot{h}\sin i}{\sin v}, \qquad -T = \dot{U} - V\frac{G}{r^2}, \qquad R = \frac{\mu}{r^2} + \dot{V} + U\frac{G}{r^2},$$

from which R, T, K obtain the values stated in the enunciation. This completes the verification required.

§ 2. I come now to another formula, which is not given with a clear explanation in regard to the arbitrariness of μ_1, μ_2 in any of the books with which I am acquainted.

Let three particles of masses C, C_1, C_2 be moving subject to their mutual gravitation. Refer the motion of C_1 to C; let the Delaunay elements of the instantaneous ellipse in this motion, as explained above

in § 1, be l_1, g_1, h_1, L_1, G_1, H_1, where $L_1 = (\mu_1 a_1)^{\frac{1}{2}}$, with arbitrary μ_1, etc. Refer the motion of C_2 to the centre of mass of C and C_1; let the Delaunay elements of the instantaneous ellipse in this motion be l_2, g_2, h_2, L_2, G_2, H_2, with $L_2 = (\mu_2 a_2)^{\frac{1}{2}}$, etc., with quite arbitrary μ_2. Put

$$m_1 = C_1\left(1+\frac{C_1}{C}\right)^{-1}, \qquad m_2 = C_2\left(1+\frac{C_1}{C}\right)\left(1+\frac{C_1+C_2}{C}\right)^{-1},$$

which reduce to C_1, C_2 respectively if we neglect C_1/C and $(C_1-C_2)/C$. Put, further,

$$F = m_1\left(\frac{\mu_1^2}{2L_1^2}+\frac{C+C_1-\mu_1}{r_1}\right)+m_2\left(\frac{\mu_2^2}{2L_2^2}+\frac{C[1+C_2(C+C_1)^{-1}]-\mu_2}{r_2}\right)$$
$$+m_1m_2\left(1+\frac{C_1+C_2}{C}\right)\left\{\frac{1}{\Delta}-\left(1+\frac{C_1}{C}\right)^{-1}\frac{1}{\kappa}\left(\frac{1}{r_2}-\frac{1}{\Delta_{02}}\right)\right\},$$

where

$$\kappa = C_1/(C+C_1),$$
$$\Delta_{02} = (r_2^2+2\kappa r_2 r_1\cos\psi+\kappa^2 r_1^2)^{\frac{1}{2}},$$
$$\Delta = [r_2^2-2r_2r_1\cos\psi+r_1^2+2\kappa r_1(r_2\cos\psi-r_1)+\kappa^2 r_1^2]^{\frac{1}{2}},$$

ψ being the angle between the directions of r_1 and r_2.

Then the equations of motion of the three masses C, C_1, C_2, are the twelve indicated by

$$m_1\frac{dL_1}{dt} = \frac{\partial F}{\partial l_1}, \quad m_1\frac{dl_1}{dt} = -\frac{\partial F}{\partial L_1}, \quad \ldots, \quad m_2\frac{dH_2}{dt} = \frac{\partial F}{\partial h_2}, \quad m_2\frac{dh_2}{dt} = -\frac{\partial F}{\partial H_2}.*$$

The proposition is well known save for the arbitrariness of μ_1, μ_2, which I have not seen recognised clearly. In order that the proof may be open to criticism, it seems necessary to give a bare statement of the steps of this proof:

(1) If H be a (regular) function of $x_1 \ldots x_n$, $y_1 \ldots y_n$, t, the solutions of the n differential equation pairs

$$\dot{x}_s = \frac{\partial H}{\partial y_s}, \qquad \dot{y}_s = -\frac{\partial H}{\partial x_s}, \qquad (s = 1 \ldots n),$$

lead to an identity

$$\Sigma y_s dx_s - H dt = d\Psi + \sum_1^{2n} B_m d\alpha_m,$$

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* *Cf.* Charlier, *Mechanik des Himmels*, **1**, 1902, 255, etc., where l might seem stated to be a linear function of the time; Poincaré, *Méth. Nouv.*, **1**, 1892, chap. i.; Tisserand, *loc. cit.*, chap. iv.; Hill, *Math. Works*, **4**, 1907, 171, 263. If we take $\mu_1 = C+C_1$, $\mu_2 = C[1+C_2(C+C_1)^{-1}]$, and suppose $\varkappa$ or $C_1/(C+C_1)$ to be small, we obtain

$$F = m_1\frac{\mu_1^2}{2L_1^2}+m_2\frac{\mu_2^2}{2L_2^2}+m_1m_2\left\{\frac{1}{(r_2^2-2r_1r_2\cos\psi+r_1^2)^{\frac{1}{2}}}-\frac{r_1\cos\psi}{r_2^2}\right\}.$$

In general, the components of angular momentum of the system about the common centre of mass are

$$\sum_1^2 m_r(G_r^2-H_r^2)^{\frac{1}{2}}\sin h_r, \qquad -\sum_1^2 m_r(G_r^2-H_r^2)^{\frac{1}{2}}\cos h_r, \qquad \sum_1^2 m_r H_r.$$

where $\alpha_1, \ldots, \alpha_{2n}$ are arbitrary constants, the $B_1, \ldots, B_{2n}$ are functions of these constants, without t, and Ψ is a function of $\alpha_1 \ldots \alpha_{2n}, t$, that is, of $x_1 \ldots y_n, t$.

Conversely, if $x_1 \ldots x_n, y_1 \ldots y_n$ be such independent functions of $\alpha_1 \ldots \alpha_{2n}, t$ as satisfy identically this differential relation, then these furnish a solution of the $2n$ given differential equations.

(2) A solution of these $2n$ differential equations, involving $2n$ arbitrary constants, which we find it convenient now to denote by $x_1' \ldots x_n'$, $y_1' \ldots y_n'$, can be obtained if we can find a solution, $S(t x_1, \ldots, x_n, y_1', \ldots, y_n')$, of the partial differential equation

$$\frac{\partial S}{\partial t} + H\left(t, x_s, \frac{\partial S}{\partial x_s}\right) = 0$$

(wherein the function H is obtained from the original H by replacing $y_1 \ldots y_n$ respectively by $\partial S/\partial x_1, \ldots, \partial S/\partial x_n$), which involves n arbitrary constants, $y_1', \ldots, y_n'$, other than the additive constant which may be attached to S. The $x_1 \ldots x_n, y_1 \ldots y_n$ required to solve the original $2n$ differential equations are then to be obtained by solution of the $2n$ equations

$$y_s = \frac{\partial S}{\partial x_s}, \qquad x_s' = \frac{\partial S}{\partial y_s'}.$$

This step (2) is clear from (1), since these last equations give

$$\Sigma y_s dx_s - H dt = dS - \Sigma x_s' dy_s'.$$

(3) The equations, with S determined as above,

$$y_s = \frac{\partial S}{\partial x_s}, \qquad x_s' = \frac{\partial S}{\partial y_s},$$

when regarded as equations of transformation from variables $x_1 \ldots x_n$, $y_1 \ldots y_n$ to variables $x_1' \ldots x_n', y_1' \ldots y_n'$, reduce a set of $2n$ differential equations which have the forms

$$\dot{x}_s = \frac{\partial}{\partial y_s}(H+F), \qquad \dot{y}_s = -\frac{\partial}{\partial x_s}(H+F),$$

to a set of $2n$ differential equations having the forms

$$\dot{x}_s' = \frac{\partial F}{\partial y_s'}, \qquad \dot{y}_s' = -\frac{\partial F}{\partial x_s'}.$$

For the identity

$$\Sigma y_s dx_s - H dt = dS - \Sigma x_s' dy_s'$$

is the same as

$$\Sigma y_s dx_s - (H+F) dt = d(S - \Sigma x_s' y_s') + \Sigma y_s' dx_s' - F dt,$$

so that either of the identities

$$\Sigma y_s' dx_s' - F dt = d\Omega + \Sigma B_q' d\alpha_q'$$
$$\Sigma y_s dx_s - (H+F) dt = d(S - \Sigma x_s' y_s' + \Omega) + \Sigma_q B' d\alpha_q'$$

involves the other.

We can therefore solve a set of differential equations

$$\dot{x}_s = \frac{\partial}{\partial y_s}(\mathrm{H}+\mathrm{F}), \qquad \dot{y}_s = -\frac{\partial}{\partial x_s}(\mathrm{H}+\mathrm{F}),$$

by a succession of steps, provided we can solve the corresponding partial equation at each step.

(4) We can reduce the solution of the partial differential equation to two (or more) simpler equations provided the function H (t, x_s, y_s), from which the partial differential equation is formed, breaks up into a sum of two parts in regard to the $y_1 \ldots y_n$, each part involving only the corresponding $x_1 \ldots x_n$, together with a mixed part, involving possibly all of $x_1 \ldots x_n$ but none of $y_1 \ldots y_n$.

Let such a form of H be that given by

$$\mathrm{M} = \mathrm{H}(t, x_1 \ldots x_r, y_1 \ldots y_r)+\mathrm{K}(t, \xi_1 \ldots \xi_s, \eta_1 \ldots \eta_s)+\mathrm{T}(t, x_1 \ldots x_r, \xi_1 \ldots \xi_s),$$

where $\xi_1, \ldots, \xi_s$ is another notation for $x_{r+1}, \ldots, x_n$ and $\eta_1, \ldots, \eta_s$ another notation for $y_{r+1}, \ldots, y_n$; and suppose that $f(t, x_1 \ldots x_r, b_1 \ldots b_r)$ and $\phi\ (t, \xi_1 \ldots \xi_s, \beta_1 \ldots \beta_s)$, with $b_1 \ldots b_r, \beta_1 \ldots \beta_s$ arbitrary constants, satisfy respectively

$$\frac{\partial f}{\partial t}+\mathrm{H}\Big(t, x_1 \ldots x_r, \frac{\partial f}{\partial x_1} \ldots \frac{\partial f}{\partial x_r}\Big) = \rho(t, x_1 \ldots x_r, b_1 \ldots b_r),$$

$$\frac{\partial \phi}{\partial t}+\mathrm{K}\Big(t, \xi_1 \ldots \xi_s, \frac{\partial \phi}{\partial \xi_1} \ldots \frac{\partial \phi}{\partial \xi_s}\Big) = \sigma(t, \xi_1 \ldots \xi_s, \beta_1 \ldots \beta_s).$$

Then, as $\partial\phi/\partial x_1 \ldots \partial\phi/\partial x_r, \partial f/\partial\xi_1 \ldots \partial f/\partial\xi_s$, as well as $\partial\phi/\partial y_k, \partial f/\partial\eta_k$, are all zero, it is clear that the differential equation

$$\frac{\partial \mathrm{S}}{\partial t}+\mathrm{H}\Big(t, x_1 \ldots x_r, \frac{\partial \mathrm{S}}{\partial x_1} \ldots \frac{\partial \mathrm{S}}{\partial x_r}\Big)+\mathrm{K}\Big(t, \xi_1 \ldots \xi_s, \frac{\partial \mathrm{S}}{\partial \xi_1} \ldots \frac{\partial \mathrm{S}}{\partial \xi_s}\Big)$$
$$= \rho(t, x_1 \ldots x_r, b_1 \ldots b_r)+\sigma(t, \xi_1 \ldots \xi_s, \beta_1 \ldots \beta_s)$$

is solved by

$$\mathrm{S} = f+\phi.$$

Now, by means of the equations

$$y_h = \frac{\partial f}{\partial x_h}, \qquad a_h = \frac{\partial f}{\partial b_h}, \qquad \eta_k = \frac{\partial \phi}{\partial \xi_k}, \qquad \alpha_k = \frac{\partial \phi}{\partial \beta_k}, \qquad \Big(\begin{matrix} h = 1 \ldots r \\ k = 1 \ldots s \end{matrix}\Big),$$

which are the same as

$$y_h = \frac{\partial \mathrm{S}}{\partial x_h}, \qquad a_h = \frac{\partial \mathrm{S}}{\partial b_h}, \qquad \eta_k = \frac{\partial \mathrm{S}}{\partial \xi_k}, \qquad \alpha_k = \frac{\partial \mathrm{S}}{\partial \beta_k},$$

define a change of variables from

$$x_1 \ldots x_r, \quad \xi_1 \ldots \xi_s, \quad y_1 \ldots y_r, \quad \eta_1 \ldots \eta_s,$$

to

$$a_1 \ldots a_r, \quad \alpha_1 \ldots \alpha_s, \quad b_1 \ldots b_r, \quad \beta_1 \ldots \beta_s.$$

We easily find, then, the identity

$$\Sigma y_h dx_h + \Sigma \eta_k d\xi_k - M dt = d[S - \Sigma a_h b_h - \Sigma \alpha_k \beta_k] + \Sigma b_h da_h + \Sigma \beta_k d\alpha_k - (T+\rho+\sigma)dt,$$

where M, T, ρ, σ stand for the functions previously referred to. This shows that the change of variables reduces the $2n$ differential equations

$$\dot{x}_h = \frac{\partial M}{\partial y_h}, \qquad \dot{\xi}_k = \frac{\partial M}{\partial \eta_k}, \qquad \dot{y}_h = -\frac{\partial M}{\partial x_h}, \qquad \dot{\eta}_k = -\frac{\partial M}{\partial \xi_k}$$

to the forms

$$\dot{a}_h = \frac{\partial F}{\partial b_h}, \qquad \dot{\alpha}_k = \frac{\partial F}{\partial \beta_k}, \qquad \dot{b}_h = -\frac{\partial F}{\partial a_h}, \qquad \dot{\beta}_k = -\frac{\partial F}{\partial \alpha_k}$$

with

$$F = T(t, x_1 \ldots x_r, \xi_1 \ldots \xi_s) + \rho(t, x_1 \ldots x_r, b_1 \ldots b_r) + \sigma(t, \xi_1 \ldots \xi_s, \beta_1 \ldots \beta_s),$$

in which, for $x_1 \ldots x_r, \xi_1 \ldots \xi_s$, are to be substituted their values in terms of $a_h, b_h, \alpha_k, \beta_k$.

The proposition is essentially well known (*cf.* Charlier, *loc. cit.*, Bd. I., pp. 77 ff.; Sommerfeld, *Atombau*, 1922, p. 679), though not usually stated with the constants b, β on the right side of the partial equations for f, ϕ as here, which is important for our purpose.

(5) The partial differential equation

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial S}{\partial \phi}\right)^2 = \frac{\mu}{r} - \frac{\mu^2}{2L^2}$$

is solved by

$$S = \int_{r_1}^{r}\left(-\frac{\mu^2}{L^2} + \frac{2\mu}{r} - \frac{G^2}{r^2}\right)^{\frac{1}{2}} dr - \int_{\frac{\pi}{2}}^{\theta} (G^2 - H^2 \operatorname{cosec}^2 \theta)^{\frac{1}{2}} d\theta + H\phi,$$

whatever constants may be (L>G>H).

(6) In the elliptic motion in which, as is well known, the equation of (5) occurs, the l, g, h defined as in § 1 of this note are such that

$$l = \frac{\partial S}{\partial L}, \qquad g = \frac{\partial S}{\partial G}, \qquad h = \frac{\partial S}{\partial H},$$

and the virtual moment of the velocity of the particle, namely,

$$\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z, \quad \text{or} \quad r^2\delta r + r^2\dot{\theta} \,.\, \delta\theta + r^2 \sin^2\theta \,.\, \dot{\phi} \,.\, \delta\phi,$$

is equal to

$$\delta S - (l\delta L + g\delta G + h\delta H).$$

Thus six equations such as

$$\frac{dx}{dt} = \frac{\partial F}{\partial \dot{x}}, \qquad \frac{d\dot{x}}{dt} = -\frac{\partial F}{\partial x},$$

in which F is a function of $x, y, z, \dot{x}, \dot{y}, \dot{z}$, by change to the variables l, g, h, L, G, H, would become six equations such as

$$\frac{dl}{dt} = \frac{\partial F}{\partial L}, \qquad \frac{dL}{dt} = -\frac{\partial F}{\partial l}.$$

(7) Consider a system of particles, it will be enough to take four, of masses C, C_1, C_2, C_3, moving under their mutual gravitation ; put

$$\sigma_s = C+C_1+ \ . \ . \ . \ +C_s, \qquad \kappa_s = \frac{C_s}{\sigma_s}, \qquad m_s = C_s\frac{\sigma_{s-1}}{\sigma_s}, \qquad \sigma_0 = C\,;$$

and let (x_s, y_s, z_s) be the co-ordinates of C_s relatively to the centre of mass of C, C_1, . . ., C_{s-1}. The kinetic energy of the system, and the moment of momentum about the axis z, passing through the centre of mass of the system, are, respectively,

$$T = \tfrac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2+\dot{z}_1^2)+\tfrac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2+\dot{z}_2^2)+\tfrac{1}{2}m_3(\dot{x}_3^2+\dot{y}_3^2+\dot{z}_3^2),$$
$$R = m_1(x_1\dot{y}_1-\dot{x}_1y_1)+m_2(x_2\dot{y}_2-\dot{x}_2y_2)+m_3(x_3\dot{y}_3-\dot{x}_3y_3),$$

while the force function is

$$U = C\sum\frac{C_s}{\Delta_{0s}}+\sum_{s,\,k}\frac{C_sC_k}{\Delta_{sk}},$$

where

$$\Delta_{01} = [x_1^2+y_1^2+z_1^2]^{\frac{1}{2}}, \qquad \Delta_{02} = [(\kappa_1x_1+x_2)^2+ \ . \ . \ . \ ]^{\frac{1}{2}}, \text{ etc.}$$

Δ_{sk} being the distance between C_s and C_k. Putting

$$r_s = (x_s^2+y_s^2+z_s^2)^{\frac{1}{2}},$$

the Hamiltonian function, H, here equal to T−U, can be taken in the form

$$H = \sum\left[\tfrac{1}{2}m_s(\dot{x}_s^2+\dot{y}_s^2+\dot{z}_s^2)-m_s\frac{\mu_s}{r_s}+m_s\frac{\mu_s^2}{2L_s^2}\right]-\sum\left(\frac{m_s\mu_s^2}{2L_s^2}-\frac{m_s\mu_s}{r_s}\right)-U,$$

where the μ_s, L_s are arbitrary. In terms of polar co-ordinates we have

$$T_s = \tfrac{1}{2}m_s(\dot{x}_s^2+\dot{y}_s^2+\dot{z}_s^2) = \tfrac{1}{2}m_s(\dot{r}_s^2+r_s^2\dot{\theta}_s^2+r_s^2\sin^2\theta_s\dot{\phi}_s^2)$$
$$= \frac{1}{2m_s}\left[\dot{u}_s^2+\frac{\dot{v}_s^2}{r_s^2}+\frac{\dot{w}_s^2}{r_s^2\sin^2\theta_s}\right],$$

where

$$u_s = \frac{\partial T_s}{\partial\dot{r}_s} = m_s\dot{r}_s, \qquad v_s = \frac{\partial T_s}{\partial\dot{\theta}_s} = m_sr_s^2\dot{\theta}_s, \qquad w_s = \frac{\partial T_s}{\partial\dot{\phi}_s} = m_sr_s^2\sin^2\theta_s\dot{\phi}_s,$$

and

$$m_s(\dot{x}_s\delta x_s+\dot{y}_s\delta y_s+\dot{z}_s\delta z_s) = u_s\delta r_s+v_s\delta\theta_s+w_s\delta\phi_s.$$

Thus the equations of motion, which consist of sets such as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}_s}\right) = \frac{\partial U}{\partial x_s}, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{y}_s}\right) = \frac{\partial U}{\partial y_s}, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{z}_s}\right) = \frac{\partial U}{\partial z_s},$$

equivalent to

$$\frac{d}{dt}(x_s) = \frac{1}{m_s}\frac{\partial H}{\partial\dot{x}_s}, \qquad m_s\frac{d}{dt}(\dot{x}_s) = -\frac{\partial H}{\partial x_s}, \ . \ . \ .,$$

are sets such as

$$\dot{r}_s = \frac{\partial H}{\partial u_s}, \quad \dot{u}_s = -\frac{\partial H}{\partial r_s}, \quad \dot{\theta}_s = \frac{\partial H}{\partial v_s}, \quad \dot{v}_s = -\frac{\partial H}{\partial \theta_s}, \quad \dot{\phi}_s = \frac{\partial H}{\partial w_s}, \quad \dot{w}_s = -\frac{\partial H}{\partial \phi_s}.$$

Consider now first the equations of this form in which, instead of H, there occurs the function

$$H_0 = \sum \left\{ \frac{1}{2m_s}\left(u_s^2 + \frac{v_s^2}{r_s^2} + \frac{w_s^2}{r_s^2 \sin^2\theta_s}\right) - m_s\frac{\mu_s}{r_s} + m_s\frac{\mu_s^2}{2L_s^2} \right\}.$$

To solve these equations we should then consider the partial differential equation

$$\frac{\partial S}{dt} + \sum \left\{ \frac{1}{2m_s}\left[\left(\frac{\partial S}{\partial r_s}\right)^2 + \frac{1}{r_s^2}\left(\frac{\partial S}{\partial \theta_s}\right)^2 + \frac{1}{r_s^2 \sin^2\theta_s}\left(\frac{\partial S}{\partial \phi_s}\right)^2\right] - m_s\frac{\mu_s}{r_s} + m_s\frac{\mu_s^2}{2L_s^2} \right\} = 0,$$

which is solved by

$$S = \Sigma m_s S_s,$$

where

$$S_s = m_s\int_{r_s 0}^{r_s}\left(-\frac{\mu_s^2}{L_s^2} + \frac{2\mu_s}{r_s} - \frac{G_s^2}{r_s^2}\right)^{\frac{1}{2}} dr_s - m_s\int_{\pi/2}^{\theta_s}\left(G_s^2 - \frac{H_s^2}{\sin^2\theta_s}\right)^{\frac{1}{2}} d\theta_s + m_s H_s \phi_s.$$

The solutions of the original ordinary equations (still with H_0 in place of H) are then to be found, in terms of the arbitrary constants L_s, G_s, H_s, l_s', g_s', h_s', by means of

$$u_s = \frac{\partial S}{\partial r_s}, \quad v_s = \frac{\partial S}{\partial \theta_s}, \quad w_s = \frac{\partial S}{\partial \phi_s}, \quad l_s' = \frac{\partial S}{\partial L_s}, \quad g_s' = \frac{\partial S}{\partial G_s}, \quad L_s' = \frac{\partial S}{\partial H_s};$$

these equations are formed of sets obtained from those written, by replacing S by $m_s S_s$, the other parts of S, or $\Sigma m_s S_s$, not containing the independent variables r_s, . . ., H_s. The motion given by these equations is in an ellipse, under an attractive acceleration μ_s/r_s^2, in which

$$L_s = (\mu_s a_s)^{\frac{1}{2}}, \qquad G_s = L_s(1-e_s^2)^{\frac{1}{2}}, \qquad H_s = G_s \cos i_s,$$
$$l_s' = m_s(u_s - e_s \sin u_s), \qquad g_s' = m_s g_s, \qquad h_s' = m_s h_s,$$

where l_s or $u_s - e_s \sin u_s$ and g_s, h_s have the interpretations previously explained.

If we now adopt these quantities as variables, the equations which we originally had to solve, which are of the forms such as

$$\dot{r}_s = \frac{\partial}{\partial u_s}(H_0 - F), \text{ etc.},$$

become, by the propositions which have been given,

$$m_s \cdot \dot{l}_s = -\frac{\partial F}{\partial L_s}, \qquad \dot{L}_s = \frac{\partial F}{m_s \partial l_s}, \text{ etc.},$$

where

$$F = \Sigma m_s F_s,$$

in which

$$F_s = \frac{\mu_s^2}{2L_s^2} - \frac{\mu_s - C\sigma_s/\sigma_{s-1}}{r_s} + C\frac{\sigma_s}{\sigma_{s-1}}\left(\frac{1}{\Delta_{0s}} - \frac{1}{r_s}\right) + \frac{1}{2}\sum_{k\neq s}^{k}\frac{\sigma_s}{\sigma_{s-1}}\frac{C_k}{\Delta_{sk}}.$$

(8) When the number of masses is three we can readily reduce these forms to those given in the enunciation.

It is at once seen that the fact that the μ_1, μ_2 in this are arbitrary depends only on the accuracy of the step (4). The other steps, though it has seemed necessary to give them for the sake of clearness, are familiar. But the whole may perhaps serve the purpose of showing how the equations of motion of three bodies may be obtained in a clearer manner than in many of the books.

A New Cometary Meteor Shower (1926 *October* 9).
By W. F. Denning.

An unusual number of large meteoric fireballs appeared in the months of September and October this year. One was observed on October 9 at $22^h\ 16^m$ G.M.T. which gave a brilliant illumination of the sky and left a streak remaining visible to the eye for about thirty minutes, during which time it underwent curious changes of form and exhibited a considerable drift amongst the stars.

As soon as satisfactory observations of the meteor came to hand it became evident that its radiant was situated in Draco and that it agreed very nearly with the radiant and date computed for Giacobini's comet of 1900, which has a period of about six and a half years and is due at perihelion on 1926 December 11. There were 53 observations of the meteor, but not more than one-third were found available for ascertaining the radiant-point, which was found to be at $262°+55°$ both by myself and Mr. King. The height was 67 to 24 miles, path 67 miles, and velocity about 18 miles per second, but the latter value is uncertain.

Mr. J. P. M. Prentice, at Stowmarket, was watching for meteors on October 9, and between $20^h\ 20^m$ and $23^h\ 20^m$ G.M.T. saw 36, of which 16 conformed with a radiant at $263°+54°$ having a diffusion of six degrees. The meteors were slow. The shower was decidedly one of unusual character, for its activity far surpassed that of the normal streams which are so abundantly distributed over the firmament. Though the night was magnificently clear, a high wind prevailed and proved rather embarrassing. Mr. Prentice thought that the hourly number of meteors visible would have been fully 17 had attention been directed uninterruptedly to the sky.

The Rev. M. Davidson, writing in the *B.A.A. Journal* for 1915 April, pointed out that the comet of Giacobini (1900) passed near the earth on October 10 and might be expected to originate a meteor shower. In a subsequent number of the *Journal* I mentioned that a number of meteors had been recorded at Bristol having a radiant at $267°+49°$ during the first half of October. The accordance was not, however, of convincing character.

In 1920 October 6–9 I find that I saw five slow meteors giving a radiant $268°+53°$, which may have belonged to the comet though it

must have passed through perihelion in the previous spring. In the *Monthly Notices* for 1920 Mr. Davidson gave the computed radiant as at 251°·5+55°·9 for October 9, and the distance from the earth's orbit as 5½ millions of miles. Dr. A. C. D. Crommelin recently pointed out that, from the elements of the comet given for the present return, the two orbits actually intersect and that the radiant would be at about 265°+54°, October 10, the R.A. being uncertain to several degrees. The Rev. M. Davidson has since made another investigation and finds that intersection occurred in accordance with Dr. Crommelin's results, and that his own computation placed the radiant at 261°+53°·5. Both in date and position, therefore, the radiants derived from the comet, fireball, and meteoric showers agree quite as closely as the nature of the observations permit, and leave little, if any, doubt as to the identity of the comet with the meteoric display of October 9 last. It seems to be another case of Pons-Winnecke and supplies an additional instance of the orbit of a periodical comet being drawn by perturbations into closer touch with the earth.

A comparison of the various positions is as follows :—

		Radiant.
Dr. A. C. D. Crommelin	Comet	265°+54° (approx.)
Rev. M. Davidson	,,	261+53·5
J. P. M. Prentice	Shower, Oct. 9	263+54
A. King and W. F. D.	Fireball ,,	262+55

Mr. King at Ashby, Lincs, and myself at Bristol were making observations on the early evening of October 9, and five of the meteors we saw were directed from a radiant in Draco at about 255°+56°. Four of the meteors were bright and equal 1st mag. or ½. A meteor which I saw at $20^h\ 20^m$ on the same night was equal to Venus, moved slowly, left a bright streak, and may have been another member of the same stream, but its direction was from a region lying about 10° or 12° S. of the Draconid radiant, and I doubt whether it should be included. It is certain, however, that this cometary shower exhibits, like similar systems connected with Biela's and Pons-Winnecke's comets, a diffuse radiant. It appears that the recent Draconid shower of October 9 was not rich in the early evening, but that its intensity increased after $20^h\ 30^m$ G.M.T.

A special feature of these cometary Draconids, and one which may serve to identify them, is the dense and sometimes durable streaks or trails which the brighter ones leave in their paths. In several instances I noted this particularly, for it is an unusual incident to find slow-moving meteors with durable trails. They leave spark-trains like the Andromedids, which, however, are quickly dissipated. Mr. Prentice saw 5 bright Draconids on October 9 (including the large fireball), and 4 of these he describes as exhibiting thick trails, several persisting for longer periods than usual.

Mr. G. Shajn (*Monthly Notices*, **83**, 343) says that the radiants of meteors from periodical comets, like the Andromedids of November,

will display very considerable and increasing diffusion until their disappearance. This conclusion accords with observation, and several of the other views he has expressed seem equally well-supported on good evidence.

Large fireballs appeared in September and October this year, as under:

	G.M.T.	Mag.	Radiant.	Notes.	Obs.
1926	h m		° °		
Sept. 6	20 45	☽	62+38	Detonating, ε Perseid, Great Yorkshire fireball.	273
15	22 59	☽	150+40 140+39	Streak 15 minutes.	5
17	19 49	☽	89+41 98+43	Very long path, 520 miles.	5
18	19 1	> ♀	235+55	A September Quadrantid.	3
20	19 2	½ ☽	74+42	Long path, 320 miles.	4
Oct. 2	19 25½	= ☽	305−10	Detonating, α Capricornid.	40
9	22 16	2× ♀	262+55	Streak 30 minutes, Draconid.	35
22	22 50	> ♀	41+20 33+19	Arietid.	3

About 20 others were reported but insufficiently observed for their real paths to be determined.

Bristol:
1926 *November* 10.

Sixth Note on W. Herschel's Extensive Nebulous Fields.

By J. G. Hagen, S.J., Assoc.R.A.S.

The present set of eleven Herschel fields completes the systematic examination of his fifty-two extensive nebulosities. Among them only one more was found, in addition to the five enumerated in the Fourth Note, p. 553, where the presence of nebulous matter is directly proved by photography, viz. No. 46.

1. Table I. is constructed like those in the preceding five notes, except that the Guiding Stars are denoted by their Bonn *DM* numbers. These numbers will be more convenient for identifying the stars on the *DM* charts. The $\Delta\alpha$ and $\Delta\delta$ will locate the Guiding Star relative to the centre of Herschel's field, as in the former notes. No. 42*a* is Herschel's field as distinguished from 42*b* proposed by I. Roberts to put it "in sequence" with the other fields.

In Table II. the extension of the nebulosities along the Parallel and the Meridian is again counted from Herschel's centre, although it is not certain that it was always meant by him to be symmetrical with the place assigned. As in the preceding fields, no definite limit of these nebulosities was found either by Herschel or by the writer.

TABLE I.

H.'s No.	Centre (1925).		*BD* Guiding Star (1925).		Mag.	Δα.	Δδ.
	h m	° ′		h m	° ′			
42*a*	21 4·7	− 1 47	− 1° 4111	21 4·3	− 1 18	7·8	− 6′	+29′
42*b*	20 44·8	− 1 47	− 2° 5375	20 46·0	− 1 47	8·9	+18	0
43	20 54·4	+16 50	+16° 4425	20 56·4	+16 32	6·2	+29	−18
44	20 55·5	+43 38	+42° 3911	20 53·5	+43 8	6·7	−21	−30
45	20 58·9	− 1 28	− 1° 4095	20 59·1	− 1 13	6·8	+ 4	+15
46	20 57·8	+43 22	+42° 3911	20 53·5	+43 8	6·7	−45	−14
47	21 6·3	+14 28	+14° 4544	21 4·4	+14 22	7·7	−28	− 6
48	21 35·4	+10 26	+10° 4604	21 38·9	+10 29	6·3	+52	+ 3
49	21 48·0	+21 38	+21° 4629	21 47·0	+21 35	7·0	−13	− 3
50	22 58·6	+25 53	+25° 4860	22 57·2	+26 6	8·5	−19	+13
51	22 59·1	+25 53	+25° 4860	22 57·2	+26 6	8·5	−26	+13
52	23 1·5	+29 25	+28° 4536	23 5·9	+29 16	7·0	+59	− 8

TABLE II.

No.	Parallel.	Meridian.	Herschel's Description.	Gal. Lat.
	′	′		°
42*a*	53	70	Much affected with whitish nebulosity.	−30
42*b*	..	..		−30
43	15	86	A good deal affected.	−20
44	30	86	Faint milky nebulosity scattered over this space, in some places pretty bright.	− 2
45	24	28	Much affected with whitish nebulosity.	−30
46	34	99	Suspected nebulosity joining to plainly visible diffused nebulosity.	− 3
47	22	83	Affected.	−22
48	15	68	Much affected.	−31
49	15	78	Affected.	−26
50	15	83	Much affected.	−32
51	21	83	Affected.	−32
52	14	78	A little affected.	−29

2. The optical densities of these last eleven fields were observed on the usual scale (I = thinnest, V = thickest) as follows.

Herschel No. 42*a*.—Around the centre the density is III–IV or IV. The quadrants *sp* and *sf* have density IV–V. In the *np* quadrant the density varies between III and IV, but in the *nf* it reaches only II–III–IV.

No. 42*b*.—This field precedes No. 42*a* twenty minutes, and was photographed by I. Roberts, because he suspected a typographical error in Herschel's R.A. The observations do not seem to con-

firm the correction. The central density is only II–III and III, which is also the general density of the field, except in the *sf* quadrant where it rises to III–IV. The field can hardly be called "much affected with whitish nebulosity."

Herschel No. 43.—An error of position might be suspected in this field because its average density is only II–III increasing to III–IV in two or three spots only. Could Herschel call it "a good deal affected?" Considering that the average optical density of the entire sky is III, and that the scale numbers I and II characterise a rather clear sky, No. 43 does not seem to belong to the Herschel fields.

No. 44.—See No. 46.

No. 45.—This is a genuine Herschel field, "much affected with whitish nebulosity." Its density is all through IV–V, varying between IV and V in a few spots.

Nos. 46 and 44 form but a single continuous nebulosity that passes from the preceding edge of the "America Nebula" through the "Gulf of Mexico" and across the "Isthmus of Central America." It coincides with the star vacancies that Professor Max Wolf exhibits on a plate in the *A.N.*, 223 (1925), 96. Field 44 represents the "Gulf," while No. 46 falls on the "Isthmus," but still within the star vacancies pointed out by Wolf.

In our 16-inch refractor neither the "Isthmus" nor any other part of the "America Nebula" is visible, any better than the fine fringes near N.G.C. 2024 and 6960, which are discussed in our *Miscellanea Astr.*, **3**, No. 65.

The photographic pictures of the "America Nebula," made by Wolf (in *Heidelberg-Publ.*, **1**), by Roberts (in his vol. **2**, p. 155) and others, prove that Herschel's field 44 is devoid of bright nebulæ, but that 46 is crossed by a narrow fringe of faint nebulous matter.

The density of this nebulosity is almost uniformly V or IV–V. South of, and *sf*, the places assigned by Herschel it diminishes to IV and even III. Where the nebulosity enters the "America Nebula," its density fades gradually to III, II, and I.

No. 47.—The plate of Roberts was pointed $4^m = 1°$ in advance of Herschel's place. Still the chart serves to identify the field. The centre of

Herschel's region has density IV–V, which is maintained throughout the field, except that following the centre there are some patches of density V, and a fainter region of density III–IV and III, *np* the centre.

Herschel No. 48.—This nebulosity is one of the densest, the estimates varying between V and IV–V. There is only one place of density III around a star cluster in the *sf* quadrant.

No. 49.—This field is even denser than No. 48. The scale number is V in the centre and on the two *f* quadrants, diminishing to IV–V in the southern and preceding directions.

Nos. 50 and 51 form one combined field, the central Meridian of which has an almost uniform density V, decreasing to IV–V on the *p* and *f* portions.

No. 52.—The density of this field is invariably IV–V, reaching V only in the *sp* and *sf* corners, but nowhere diminishing to III.

3. *General Conclusions* (*a*).—It will be noticed that Herschel nowhere mentions chains or rings of stars such as are described in our *Miscellanea Astronomica* and in the *Preparatory Catalogue* (Specola Vat. X.). Nor could he, because all his nebulosities extend far beyond the limits of his sweeps, whereas those chains are found only on the margins. It follows that star chains could not affect Herschel's *visual observations of stellar density within the ring*, nor lead *to the conception of a difference in background colour inside and out.*

(*b*) Though in no case did photography reproduce the visual picture of Herschel's nebulosities, the presence of nebulous matter is proved in at least six of the regions. Barnard was so sure that the same holds for the remaining fields (48 at his time) that he called it "a little unreasonable to suppose that Herschel, who made so few blunders compared with the wonderful and varied work that he accomplished, should be so palpably mistaken in 48 out of 52 observations of this kind." (*Ap. J.*, **17** [1903], 77.)

(*c*) When Herschel says: "I was not inattentive to extensive diffusions of nebulosity which occasionally fell under my observations," he implies that during the twenty years observing he made only occasional notes of these nebulosities. This explains how in seven cases he could record the same nebulosity a second time.* Nevertheless his general impression of the multitude of these nebulosities was "that the abundance of nebulous matter diffused through such an expansion of the heavens must exceed all imagination" (*Sc. Papers*, **2**, 464).

Herschel made no attempt at physical explanations of his nebulosities, except that they are often joined to real nebulæ. He thus

* The duplicate records are (2–4), (7–8), (9–10), (14–15), (20–21), (37–38), (44–46). In a systematic search these pairs could not be set down as separate nebulosities. We have therefore but 45 nebulous fields of Herschel, instead of 52, and if No. 44 is an error, we have only 44.

anticipated our repeated statement that nebulæ are more abundant where nebulosities are denser, and then preferably on the borders.

(*d*) Is it credible that, after a systematic examination of all the 52 regions, we in Rome should see something that Herschel did not see, and that Herschel should have seen something that we cannot see ? If so, then Herschel's visual nebulosities must be rejected as not seen again by anyone within more than a century.

Everyone may now form his own opinion as to whether Herschel's nebulosities are but *effects of the uneven distribution of stars brighter than the fifteenth magnitude* or, on the contrary, real nebulous matter, whatever its nature may be.

Corrections to the Six Notes.

1. First Note, vol. **86**, p. 145, No. 1, the galactic latitude should be $-51°$ instead of $-50°$.

2. Third Note, p. 439, Table II., No. 14, *read* $32'$ *instead of* $62'$.

3. Third Note, p. 440, No. 16, the two stars, to which attention was called, were found by Madame I. Roberts to be defects on the original negative.

4. Fifth Note, p. 642, Table I., No. 33, under Guiding Star *read* $16^{m}{\cdot}3$ *instead of* $14{\cdot}3$, and $\Delta\alpha = +49'$ *instead of* $+19'$.

Vatican Observatory :
1926 *October.*